FEM/BEM methods for Signorini-type problems — error analysis, adaptivity, preconditioners

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Chapter 1

Introduction

Finite element methods are well known for the approximation of solutions of partial differential equations using a variational approach for problems arising in mathematical physics and engineering. Boundary element methods have become more and more a useful alternative to finite element methods with some advantages in special situations. Obstacle problems and Signorini problems arise, e.g., from contact problems in linear elasticity, membrane theory, etc. For an overview, we refer to the monographs of Glowinski [39], Glowinski, Lions, and Trémolières [40] and Hlaváček, Haslinger, Nečas, and Lovíšek [56]. These monographs treat variational inequalities and their discretizations in a classical way using finite element methods, often applying solution procedures originating in optimization theory, c.f., e.g., Ciarlet [25]. A more modern approach can be found in Kornhuber [63], involving multigrid methods and hierarchical error estimators for finite element discretizations of variational inequalities.

Beginning with the paper of Hsiao, Wendland [60] there is a vast amount of literature dealing with error analysis for the $h$, $p$ and $hp$ versions, regularity, error estimators and adaptive schemes, fast solvers and efficient preconditioners for boundary element methods. The boundary element method for variational inequalities is used only in a few cases, e.g., Gwinner, Stephan [48] and Spann [87], treating the equivalence to the original formulation and the norm convergence for a quasi-uniform $h$ version with low-order polynomials for the Laplace and Lamé operators based on an abstract convergence proof, e.g., from [39, Theorem 1.5.2].

The symmetric coupling of finite elements and boundary elements was done first in Costabel, Stephan [29], [30]. A formulation for an interface problem with Signorini-type interface conditions was analyzed in Carstensen, Gwinner [23]. Dual formulations for the symmetric coupling procedure with mixed finite elements have been analyzed in Meddahi, Valdés, Menéndez, Pérez [77]. Error estimators for mixed finite elements have been analyzed by Carstensen [20] and for the symmetric coupling by Gatica, Meddahi [38] and Carstensen, Funken [22].

In all problems containing Signorini conditions the boundary element method is incorporated by using the Steklov-Poincaré operator, i.e., the Dirichlet to Neumann mapping, or its inverse. Because the Steklov-Poincaré operator cannot be discretized analogously to the standard integral operators $V, K, K', W$, it has to be discretized by computing a Schur complement of discretized operators. This Schur complement has the advantage to correspond to a symmetric, positive definite operator. For this advantage one has to pay by an additional complexity in the error analysis and, if the Schur complement has to be performed explicitly in the numerical computation, by additional computational costs. Therefore we have to design our methods in such a way that any additional costs
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will be minimized.

Our main concern in this work is the construction of efficient and robust schemes for the solution of Signorini problems. An "efficient" scheme mainly means a scheme with a high convergence rate, better than the convergence rate of a quasi-uniform $h$ version, i.e., an adaptive mesh-refining algorithm (cf. [21]) or a $p$ or $hp$ version. But this also means efficiency in terms of computing time, e.g., the time for computing the Galerkin matrix and for the solution of the linear system have to be comparable and small. This immediately demands an iterative solver with a good preconditioner. Preferable are solvers which allow the use of preconditioners known from standard variational formulations, e.g., multiplicative and additive Schwarz methods. Robustness means usually that we do not need to know much details of the problem (e.g., the location of singularities), or that small numerical errors, e.g., in the numerical quadrature or rounding errors in the solver, will not cause trouble. In the case of Signorini problems, which are related to fracture or crack problems, we often are also interested in an improved accuracy of the traction or gradient of the solution. There are some post-processing schemes available, but the dual methods introduce the gradient as an independent variable, such that there is no need to reconstruct the gradient from the solution itself and therefore losing an order of accuracy.

In this work we first extend the approach of Gwinner, Stephan [48] to a true $p$ version based on Lagrange polynomials to Gauss-Lobatto nodes and show convergence for the $p$ version and for an $h$ version with arbitrary high polynomials (Theorem 3.2). Based on two-level decomposition results (Mund, Stephan [78] and Heuer, Mellado, Stephan [54]) we prove hierarchical error estimators (see Bank, Smith [7] and Bank [8]) for two-level decompositions based on local subspace refinements and on bubble functions (Theorem 3.6, Theorem 3.7). The Steklov-Poincaré operator is eliminated from the computation of the error estimators by using a decomposition argument proved in Lemma 2.5. Then, we have to solve a global auxiliary variational inequality, contrary to the usual one dimensional subspace problems, because we have no longer Galerkin orthogonality. We also describe additive and multiplicative Schwarz operators for the Steklov-Poincaré operator, which are simply modified operators corresponding to the hypersingular integral operator. These Schwarz operators are used as preconditioners for the Polyak algorithm (see [81] and [82]), which is a modified CG algorithm (Theorem 3.8, 3.9, 3.10 and 3.11). Alternatively, we have applied the monotone multigrid method from [63] to the boundary element method using the Steklov-Poincaré operator (Theorem 3.12, Algorithm 3.4).

For the symmetric FEM-BEM coupling with Signorini interface conditions [23], firstly, we give a proof for the quasi-uniform $hp$ version with a linear finite element part (Theorem 4.4), which was not covered directly by [23]. Then, we prove a hierarchical error estimator based on local subspace refinements for the fully non-linear case (Theorem 4.6). The proof is based on the construction of a series of auxiliary variational inequalities and considerably more involved, than, e.g., the proof in [78], due to the lack of Galerkin orthogonality. A main topic in the design of the auxiliary problems was also the computability of the resulting error estimators. The solution of a global problem can only be eliminated for variables which are not the subject of the Signorini condition. Here we obtain local indicators for the finite element part and a global auxiliary variational inequality only involving the boundary element part. Later, we show how to construct preconditioners for this kind of block-structure in the linear case, based on standard preconditioners (Theorem 4.7).

Next, we give a dual formulation for the linear FEM-BEM coupling with Signorini-
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type interface conditions using the inverse Steklov-Poincaré operator. We show the equivalence of the minimization problem, the saddle point problem and the variational inequality with two Lagrange multipliers to the original problem (Theorem 5.5). One Lagrange multiplier describes the solution, the second Lagrange multiplier incorporates the Signorini condition. For the discretized version we show an abstract a priori error estimate (Theorem 5.8), and then for a special choice of discrete subspaces we extend the Babuška-Brezzi result from [3] to three dimensions. Assuming enough regularity we obtain a linear convergence rate with $h$ (Theorem 5.9). A suitable solver is an Uzawa-type algorithm, such that the interior iteration consists of solving a well-known system, for which an efficient solving strategy exists [38]. Finally, we prove a residual based error indicator for our dual-mixed FEM-BEM coupling scheme with Signorini-type interface conditions (Theorem 5.10 and Theorem 5.11).

For the hypersingular integral equation in two dimensions using an algebraically graded mesh we prove the properties of the multilevel additive Schwarz operator utilizing the results in [96] for the multilevel multiplicative Schwarz operator (Lemma 6.1 and Lemma 6.3). For the quasi-uniform $hp$ version in three dimensions for the weakly singular integral operator we extend the approach of [74] for the 2-level multiplicative Schwarz operator and we prove that the contraction rate is independent of $h$ and depends only logarithmically on $p$ (Theorem 6.2).

Finally, we use a regularity result of [57], which states the solution of the hypersingular integral equation on a polyhedral domains to be in a special countably normed space, defined on the surface of the domain. We prove exponentially fast convergence of the Galerkin method for an $hp$ version with a geometrically refined mesh, by constructing a spline, approximating the solution and proving exponentially fast convergence in the $H^{1/2}(\Gamma)$ and the $\tilde{H}^{1/2}(\Gamma)$ norm (Theorem 7.4 and Theorem 7.5). Our approach here is the generalization of Guo, Babuška [46] to highly anisotropic meshes and weight functions describing edge and corner-edges singularities. We have to decompose the proof into several parts. The main difficulty is the non-locality of the fractional Sobolev norms involved. Therefore, we have to decompose the function into a finite linear combination and have to approximate each term separately either by using the trace theorem (near the border) or by using real-interpolation results.

There have been several approaches for exponential fast convergence in three dimensions, e.g., Elschner [35] for a second kind integral equation and Guo [45] and Guo, Babuška [43], [44], [42]. Here we use a combination of interpolation and trace theorems. This results extends our work in [89, 70, 58, 53, 57, 71].

In the following we want to outline the contents of the chapters of this work.

In Chapter 2 we present some well-known results on the solvability of variational inequalities and define the Sobolev spaces used in this thesis. Further, in Lemma 2.5 we prove a decomposition result which allows us to prove some hierarchical error estimators. Finally, we present some general solution algorithms for variational inequalities based on bilinear forms, e.g., the relaxation method (SOR) with subspace projection [39] and the preconditioned Polyak algorithm [81, 82] for positive definite and symmetric problems, and an Uzawa-like algorithms for saddle point problems [25].

In Chapter 3 we give the formulation of the Signorini contact problem using the symmetric BEM for the Laplacian and the Lamé operator. We extend the arguments of [48] to cover also the $p$ version. Then, we use the abstract result of Chapter 2 to prove a posteriori error estimates for the $h$ version based on local subspace enrichment by mesh refinement and by adding bubble functions. The latter a posteriori error estimate is also extended to the $p$ and $hp$ versions (cf. [54]). Then we present some results on precondi-
tioners for the Polyak algorithm suitable for the discrete Steklov-Poincaré operator. We also apply the monotone multigrid algorithm to our boundary element method (cf. [63]). In Chapter 4 we investigate the FEM-BEM coupling with Signorini contact within the abstract framework given in [23]. First, we also cover the \( p \) version in the linear case using the arguments in Chapter 3. Then, we prove a hierarchical a posteriori error estimate for the non-linear case. For the linear case we show how standard preconditioners can be used to construct an efficient preconditioner for the Polyak algorithm applied to the symmetric FEM-BEM coupling with Signorini contact. In Chapter 5 we reformulate the linear version of the problem in [23] into a dual-mixed formulation and show equivalence of both formulations. This dual-mixed formulation treats the stress, or respectively the gradient, by an independent variable which is discretized using Raviart-Thomas elements [84]. For this formulation and its discretization we show existence and uniqueness by applying the abstract framework of [56] and using a discrete inf-sup condition, which was proved for the 2d case in [3] and is extended here to the 3d case using the arguments in [2]. For this dual-mixed formulation we propose an Uzawa-like algorithm which allows the use of fast solvers [38] suitable for the standard dual-mixed FEM-BEM coupling. For our dual-mixed FEM-BEM coupling with Signorini contact we also prove a residual-type a posteriori error estimate, generalizing the results in [20, 38].

In Chapter 6 we now investigate preconditioners for boundary element methods with structured grids. Firstly, we prove that the multilevel additive Schwarz method applied to the hypersingular integral equation on an algebraically graded mesh in two dimensions depends at most logarithmically on the number of degrees of freedom. Secondly, we prove that the multiplicative Schwarz method applied to the weakly singular integral equation on an quasi-uniform mesh in three dimensions depends only logarithmically on the polynomial degree and is independent on the number of mesh elements.

In Chapter 7 we use a regularity result from [57] for the solution of the hypersingular integral equation on polyhedral domains with piecewise polynomial data. We show that functions in this countably normed space, used in the regularity result, can be approximated with an exponentially fast convergence rate on a tensor product mesh, using geometric refinement towards the edges and linearly increasing polynomial degrees. This chapter can be regarded as the conclusion our series of papers [89, 70, 58, 53, 57, 71] dealing with regularity results and approximation theorems in three dimensions. Here we use a combination of interpolation by the real K-method [9] and trace theorems [1]. The method of construction is a generalization of [46] towards anisotropic meshes. The computations have been performed with the software package \textit{maiprogs}, dedicated to numerical experiments in BEM and FEM (see [69] and [68]).

Throughout the work, \( C, c, C_1 \), etc. denote generic positive constants which do not depend on essential parameters like the mesh size \( h \) or the polynomial degree \( p \) if not stated otherwise.

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Chapter 2

Variational inequalities

2.1 Introduction - abstract theory

Variational inequalities arise in the context of contact problems or in the description of non-linear material behavior. This work deals with the use of BEM alone or of FEM-BEM coupling methods for primal or dual formulations. Common to all problems handled here is a domain or part of a domain with constant material coefficients, therefore a fundamental solution exists. If the domain is split into two parts, one with constant material coefficients and the second with space dependent or even non-linear material coefficients, we use a FEM-BEM coupling procedure.

Two different cases of variational inequalities have to be distinguished:

In the first case there exists an elliptic (and symmetric) bilinear form $a(\cdot, \cdot)$ and a linear functional $l(\cdot)$ on a convex $K \subset H$, where $H$ is a Hilbert space. Then, the variational inequality reads:

There exists $u \in K$ such that

$$a(u, v - u) \geq l(v - u) \quad \forall v \in K.$$  

In the second case, arising from dual formulations, there exists an elliptic (and symmetric) bilinear form $a(\cdot, \cdot) : H \times H \to \mathbb{R}$, a continuous bilinear form $b(\cdot, \cdot) : V \times H \to \mathbb{R}$ which fulfills an inf-sup condition and linear functionals $r(\cdot) : H \to \mathbb{R}$, $l(\cdot) : V \to \mathbb{R}$, and a convex $K \subset V$, where $V$ is a second Hilbert space. Then the variational inequality reads:

There exists $(q, u) \in H \times V$ such that

$$a(q, p) + b(p, u) = r(p) \quad \forall p \in H,$$

$$b(q, v - u) \geq l(v - u) \quad \forall v \in K.$$  

In this work we analyze uniqueness and existence of solutions arising from various problems, prove the properties of fast solvers and preconditioners and investigate a posteriori error estimates and adaptive schemes.

2.1.1 Solvability of variational inequalities

Definition 2.1 Let $K \subset H$ be a convex finite dimensional subset of a Hilbert space $H$ with norm $\| \cdot \|$. Let $a(\cdot, \cdot)$ be a coercive and continuous bilinear form and $l(\cdot)$ a linear functional. The variational inequality is given by:

Find $\bar{u} \in K$ such that

$$a(\bar{u}, v - \bar{u}) \geq l(v - \bar{u}) \quad \forall v \in K.$$  

(2.1)
In [39], for example, we find that this kind of problem is uniquely solvable. Useful for estimating the error of an approximate solution will be the following lemma.

**Lemma 2.1** Let $K \subset H$ be a convex finite dimensional subset of a Hilbert space $H$. Let $a(\cdot, \cdot)$ be a coercive, continuous and symmetric bilinear form and $l(\cdot)$ a linear functional. Let $J(v) := \frac{1}{2}a(v, v) - l(v)$ for all $v \in K$. Let $\hat{u}$ be the solution of (2.1). Then there holds

$$a(v - \hat{u}, v - \hat{u}) \leq 2J(v) - 2J(\hat{u}) \quad \forall v \in K. \quad (2.2)$$

**Proof.** Due to (2.1) there holds

$$2J(v) - 2J(\hat{u}) = a(v, v) - a(\hat{u}, \hat{u}) - 2l(v - \hat{u}) \geq a(v, v) - a(\hat{u}, \hat{u}) - 2a(\hat{u}, v - \hat{u}) = a(v, v) - 2a(\hat{u}, v) + a(\hat{u}, \hat{u}).$$

By the symmetry of $a(\cdot, \cdot)$ the assumption follows. \hfill $\square$

**Remark 2.1** The proof above shows that (2.2) fails if $a(\cdot, \cdot)$ is non-symmetric.

Additional difficulties arise from the discretization of (2.1) by the approximation of $K$ and in case that the bilinear form $a(\cdot, \cdot)$ can only be computed approximately. Frequently, the bilinear form $a(\cdot, \cdot)$ is a Schur complement $S$ of operators $A, B, C, D$. The canonical discretization is usually not possible, therefore $S$ has to approximated by the Schur complement of the discretization of $A, B, C, D$. In the following we treat this in an abstract setting (closely following [18, Section 2.5]) and also discuss the error estimation in the context of a variational inequality defined by a Schur complement. Let the Banach spaces $X, Y$ be reflexive with the dual spaces $X^*, Y^*$. Let $X_h, Y_h$ be finite dimensional subspaces of $X, Y$. We denote the canonical embeddings by

$$i_h : \begin{cases} X_h \to X \\ x \mapsto x \end{cases}, \quad j_h : \begin{cases} Y_h \to Y \\ y \mapsto y \end{cases}$$

with their duals

$$i_h^* \in L(X^*, X_h^*), \quad j_h^* \in L(Y^*, Y_h^*).$$

Additionally, we consider some linear operators and their discretizations

$$A \in L(X, X^*), \quad A_h := i_h^* Ai_h \in L(X_h, X_h^*),$$

$$B \in L(Y, X^*), \quad B_h := i_h^* Bj_h \in L(Y_h, X_h^*),$$

$$C \in L(X, Y^*), \quad C_h := j_h^* Ci_h \in L(X_h, Y_h^*),$$

$$D \in L(Y, Y^*), \quad D_h := j_h^* Dj_h \in L(Y_h, Y_h^*).$$

Further, we assume that $D$ is strongly positive, i.e., there exists $\alpha_D > 0$ such that

$$D(y)(y) \geq \alpha_D \|y\|^2_Y \quad \forall y \in Y.$$  

According to the Lax-Milgram lemma, $D$ and $D_h$ are continuously invertible,

$$D^{-1} \in L(Y^*, Y) \quad \text{and} \quad D_h^{-1} \in L(Y_h^*, Y_h).$$
Now, we can define the Schur complement, the discretization of the Schur complement and its discrete approximation
\[
S := A + BD^{-1}C \in L(X, X^*), \\
\tilde{S}_h := \frac{i_h^*}{i_h^*} S i_h \in L(X_h, X_h^*), \\
S_h := A_h + B_h D_h^{-1} C_h \in L(X_h, X_h^*).
\]

Note that in general we have \(S_h \neq \tilde{S}_h\). In contrast to \(\tilde{S}_h\), the discrete approximation \(S_h\) can be calculated numerically by computing the discrete operators \(A_h, B_h, C_h, D_h\) and solving a system of linear equations involving \(D_h\). \(\tilde{S}_h\) still involves the computation of \(D^{-1}\) which is in general not possible. But \(\tilde{S}_h\) is the operator which is consistent with the discretization of the original problem. Therefore it remains to estimate \(\|S_h - \tilde{S}_h\|\).

**Definition 2.2** For \(y \in Y\) define
\[
\text{dist}(y, Y_h) := \text{dist}_Y(y, Y_h) := \inf_{y_h \in Y_h} \|y - y_h\|_Y.
\]

**Lemma 2.2** [18, Lemma 2.4] For all \(x_h \in X_h\) and \(x \in X\) there holds
\[
\| (\tilde{S}_h - S_h)x_h \|_{X_h^*} \leq c_1 \cdot \text{dist}_Y(D^{-1}Cx_h, Y_h) \\
\leq c_1 \cdot \text{dist}_Y(D^{-1}Cx, Y_h) + c_2 \cdot \|x - x_h\|_X \tag{2.3}
\]
and
\[
\| S_h x - i_h^* S x \|_{X_h^*} \leq c_1 \cdot \text{dist}_Y(D^{-1}Cx, Y_h) + (c_2 + c_3) \cdot \|x - x_h\|_X. \tag{2.4}
\]
The constants \(c_1 := \frac{\|B\|\|D\|}{\alpha D}, c_2 := c_1 \cdot \|C\|/\alpha_D\) and \(c_3 := \|A\| + \|B\| \cdot \|C\|/\alpha_D\) depend only on the norms of \(A, B, C, D\) and on \(\alpha_D\) and do not depend on the discretization.

**Lemma 2.3** Let \(B = C^*\) and let \(e_h(u) := \text{dist}_Y(D^{-1}Cu, Y_h)\). The symmetric mapping \(E_h : X \to X^*\) is defined by
\[
E_h := C^* (D^{-1} - j_h^* D ji_h) C. \tag{2.5}
\]
Then \(E_h\) is bounded and there holds
\[
0 \leq \langle E_h u, u \rangle \leq (c_1 e_h(u))^2 \quad \forall u \in X \tag{2.6}
\]
and
\[
\|E_h(u)\|_X \leq c_0 e_h(u) \quad \forall u \in X \tag{2.7}
\]
with constants \(c_0, c_1\) independent of \(h\).

**Proof.** Let \(y := Cu \in Y^*\). Since \(D\) is positive definite, there exists a solution \(z \in Y\) of \(Dz = y\) and a solution \(zh \in Y_h\) of \(j_h^* D j_h zh = j_h^* y\), i.e., \(zh\) is the Galerkin approximation of \(z \in Y\) in \(Y_h\) and we have
\[
\|z\|_Y \leq C_1 \|y\|_{Y^*} \text{ and } \|j_h zh\|_Y \leq C_2 \|y\|_{Y^*}. \tag{2.8}
\]
Then, we have
\[
\langle E_h(u), v \rangle = \langle z - j_h zh, Cv \rangle \leq (C_1 + C_2) \|y\|_{Y^*} \|Cv\|_{Y^*} \leq C_3 \|u\|_X \|v\|_X,
\]
where
\[
\|E_h(u)\|_X \leq c e_e(u).
\]
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i.e., $E_h$ is bounded. Due to the Galerkin orthogonality we have

$$0 \leq \|z - j_h z_h\|_D^2 = \|z\|_D^2 - \|j_h z_h\|_D^2$$

$$= z^* D z - (j_h z_h)^* D j_h z_h = z^* y - z_h^* j_h^* y$$

$$= y^* D^{-1} y - y^* j_h (j_h^* D j_h)^{-1} j_h^* y$$

$$= \langle E_h u, u \rangle,$$

i.e., due to the quasi-optimal error estimate we have

$$0 \leq \langle E_h u, u \rangle = \|z - j_h z_h\|_D^2 \leq (c_1 e_h(u))^2.$$  

Using the boundedness of $C$ and the quasi-optimal error estimate we have

$$\|E_h u\|_{X^*} \leq \|C\| \cdot \|z - j_h z_h\|_Y \leq c_0 e_h(u).$$

Therefore, the assertion of the Lemma follows. □

**Lemma 2.4** Let $B = C^*$ and $S, S_h$ be strongly positive and symmetric. Let $l \in X^*$. Let $a(u, v) := \langle Su, v \rangle$, $\forall u, v \in X$ and let $a_h(u, v) := \langle S_h u_h, v_h \rangle$, $\forall u_h, v_h \in X_h$. Let $J(v) := \frac{1}{2} a(v, v) - l(v)$, $\forall v \in X$ and let $J_h(v_h) := \frac{1}{2} a_h(v_h, v_h) - l(i_h v_h)$, $\forall v_h \in X_h$. Let $K_h \subseteq K \subseteq X$ closed and convex. Let $u \in K$ be the element which minimizes $J(v)$ in $K$ and let $u_h \in K_h$ be the element which minimizes $J_h(v_h)$ in $K_h$. Then there holds

$$\|u - u_h\|_S^2 \leq 2 (J_h(u_h) - J(u)) + \langle E_h i_h u_h, i_h u_h \rangle. \quad (2.9)$$

with

$$0 \leq \langle E_h i_h u_h, i_h u_h \rangle \leq c (e_h(u) + \|u - u_h\|_X)^2 \quad (2.10)$$

where $e_h(u) := \text{dist}_Y (D^{-1} Cu, Y_h)$.

**Proof.** Due to Lemma 2.1 we have

$$\|u - u_h\|_S^2 \leq 2 (J(i_h u_h) - J(u)) = 2 (J_h(u_h) - J(u)) + 2 (J(i_h u_h) - J_h(u_h)).$$

Due to Lemma 2.3 we can write

$$2 (J(i_h u_h) - J_h(u_h)) = \langle E_h i_h u_h, i_h u_h \rangle \geq 0$$

and there holds

$$\langle E_h i_h u_h, i_h u_h \rangle = \langle E_h (u - i_h u_h), (u - i_h u_h) \rangle - 2 \langle E_h u, u - i_h u_h \rangle + \langle E_h u, u \rangle$$

$$\leq \|E_h\| \cdot \|u - i_h u_h\|_X^2 + 2 c_0 e_h(u) \|u - i_h u_h\|_X + (c_1 e_h(u))^2.$$  

□

2.1.2 A posteriori error estimates and adaptive schemes

We will prove the existence of a posteriori error estimators based on the following lemma.
Lemma 2.5 (cf. [64, Proposition 4.2]) Let $\bar{K} \subset H$ be a finite dimensional convex subset of a Hilbert space $H$ with $K \subset \bar{K}$. Let $a$ and $\bar{a}$ be symmetric, positive definite and continuous bilinear forms. Let $\bar{u}$ be an approximate solution of (2.1). The discrete defect problem is given by:
Find $e \in \bar{K}$ such that
\[
a(e, v - e) \geq l(v - e) - a(\bar{u}, v - e) \quad \forall v \in \bar{K} - \bar{u}. \tag{2.11}
\]
The preconditioned defect problem is given by:
Find $\tilde{e} \in \bar{K}$ such that
\[
\bar{a}(\tilde{e}, v - \tilde{e}) \geq l(v - \tilde{e}) - a(\bar{u}, v - \tilde{e}) \quad \forall v \in \bar{K} - \bar{u}. \tag{2.12}
\]
Let
\[
\gamma_0 \bar{a}(v, v) \leq a(v, v) \leq \gamma_1 \bar{a}(v, v) \quad \forall v \in \text{span}\{e, \tilde{e}\} \tag{2.13}
\]
then there holds
\[
c_0 \|\tilde{e}\|_a^2 \leq \|e\|_a^2 \leq c_1 \|\tilde{e}\|_a^2 \tag{2.14}
\]
with $c_0 = (\gamma_0^{-1} + 2\gamma_1(1 + \gamma_0^{-1}))^{-1}$ and $c_1 = \gamma_1 + 2\gamma_0^{-1}(1 + \gamma_1)$.

**Proof.** By symmetry arguments it is sufficient to establish only the right inequality in (2.14). Inserting $v = \tilde{e}$ into the original discrete defect problem (2.11), we obtain
\[
a(e, \tilde{e} - e) \geq l(\tilde{e} - e) - a(\bar{u}, \tilde{e} - e),
\]
and, equivalently,
\[
a(e, e) \leq a(e, \tilde{e}) + l(e - \tilde{e}) - a(\bar{u}, e - \tilde{e}). \tag{2.15}
\]
Now inserting $v = e$ into (2.12) gives
\[
a(e, e) \leq a(e, \tilde{e}) + l(e - \tilde{e}) - a(\bar{u}, e - \tilde{e}) \leq a(e, \tilde{e}) + \bar{a}(\tilde{e}, e - \tilde{e}). \tag{2.16}
\]
Now the inequality $2a(e, \tilde{e}) \leq \|e\|_a^2 + \|\tilde{e}\|_a^2$ and the Cauchy-Schwarz inequality yield
\[
\|e\|_a^2 \leq \frac{1}{2} \|e\|_a^2 + \frac{1}{2} \|\tilde{e}\|_a^2 + \|\tilde{e}\|_a \|e - \tilde{e}\|_a,
\]
and, equivalently,
\[
\|e\|_a^2 \leq \|\tilde{e}\|_a^2 + 2\|\tilde{e}\|_a \|e - \tilde{e}\|_a. \tag{2.17}
\]
Inserting $v = \tilde{e}$ in (2.11) and $v = e$ in (2.12) again and adding the two resulting inequalities, we obtain
\[
a(e, \tilde{e} - e) + \bar{a}(\tilde{e}, e - \tilde{e}) \geq 0
\]
which can be reformulated to
\[
\|\tilde{e} - e\|_a^2 \leq a(\tilde{e}, \tilde{e} - e) - \bar{a}(\tilde{e}, \tilde{e} - e).
\]
Using the Cauchy-Schwarz inequality we obtain
\[
\|\tilde{e} - e\|_a^2 \leq \|\tilde{e}\|_a \|\tilde{e} - e\|_a + \|\tilde{e}\|_a \|\tilde{e} - e\|_a \leq (1 + \gamma_0^{-1}) \|\tilde{e}\|_a \|\tilde{e} - e\|_a
\]
which gives
\[
\|\tilde{e} - e\|_a \leq (1 + \gamma_0^{-1}) \|\tilde{e}\|_a.
\]
 Altogether we obtain by using (2.13) 
\[
||e||_a^2 \leq ||\tilde{e}||_a^2 + 2||\tilde{e}||_a ||e - \tilde{e}||_a \\
\leq \gamma_1 ||\tilde{e}||_a^2 + 2\gamma_1^{1/2}||\tilde{e}||_a ||e - \tilde{e}||_a \\
\leq \gamma_1 ||\tilde{e}||_a^2 + 2\gamma_1^{1/2}||\tilde{e}||_a (1 + \gamma_0^{-1})||e||_a \\
= \gamma_1 (1 + 2(1 + \gamma_0^{-1}))||\tilde{e}||_a^2.
\]

\[\square\]

**Lemma 2.6** Let \( H_1 \subseteq H_2 \) be finite dimensional subspaces of a Hilbert space \( H \) and let \( K \subseteq H_1, \tilde{K} \subseteq H_2 \) be convex subsets, not necessarily with \( K \subseteq \tilde{K} \). Let \( a \) and \( \tilde{a} \) be coercive and continuous bilinear forms, but not necessarily symmetric, i.e., there are constants \( c_1, C_1 \) and \( c_2, C_2 \) with 
\[
a(u, v) \leq C_1 ||u|| ||v|| \quad \text{and} \quad \tilde{a}(u, v) \leq C_2 ||u|| ||v|| \quad \forall u, v \in H_2
\]
and 
\[
c_1 ||v||^2 \leq a(u, u) \quad \text{and} \quad c_2 ||u||^2 \leq \tilde{a}(u, u) \quad \forall u \in H_2.
\]

Let \( \hat{u} \) be an approximate solution of (2.1). We consider the discrete defect problem: 
Find \( e \in \tilde{K} \) such that 
\[
a(e, v - e) \geq l(v - e) - a(\hat{u}, v - e) \quad \forall v \in \tilde{K} - \hat{u},
\]
and the preconditioned defect problem: 
Find \( \tilde{e} \in \tilde{K} \) such that 
\[
\tilde{a}(\tilde{e}, v - \tilde{e}) \geq l(v - \tilde{e}) - a(\hat{u}, v - \tilde{e}) \quad \forall v \in \tilde{K} - \hat{u}.
\]

Then there holds 
\[
\frac{c_1 c_2}{C_2 c_1 + C_2 C_1 + C_1^2} ||\tilde{e}|| \leq ||e|| \leq \frac{C_1 c_2 + C_2 C_1 + C_2^2}{c_1 c_2} ||\tilde{e}||.
\]

**Proof.** By symmetry arguments it is sufficient to establish only the right inequality in (2.20). Inserting \( v = \tilde{e} \) into the original discrete defect problem (2.18), we obtain 
\[
a(e, \tilde{e} - e) \geq l(\tilde{e} - e) - a(\hat{u}, \tilde{e} - e)
\]
and, equivalently, 
\[
a(e, e) \leq a(e, \tilde{e}) + l(e - \tilde{e}) - a(\hat{u}, e - \tilde{e}).
\]\[\text{(2.21)}\]

Now inserting \( v = e \) into (2.19) gives 
\[
a(e, e) \leq a(e, \tilde{e}) + l(e - \tilde{e}) - a(\hat{u}, e - \tilde{e}) \leq a(e, \tilde{e}) + \tilde{a}(\tilde{e}, e - \tilde{e}).
\]\[\text{(2.22)}\]

Now, coercitivity and continuity of \( a \) and \( \tilde{a} \) yield 
\[
c_1 ||e||^2 \leq ||\tilde{e}|| (C_1 ||e|| + C_2 ||e - \tilde{e}||).
\]\[\text{(2.23)}\]

Inserting \( v = \tilde{e} \) in (2.18) and \( v = e \) in (2.19) again and adding the two resulting inequalities, we obtain 
\[
a(e, \tilde{e} - e) + \tilde{a}(\tilde{e}, e - \tilde{e}) \geq 0
\]
which can be reformulated to
\[ a(\tilde{e} - e, \tilde{e} - e) \leq a(\tilde{e}, \tilde{e} - e) - \tilde{a}(\tilde{e}, \tilde{e} - e) \]
and, equivalently,
\[ \tilde{a}(\tilde{e} - e, \tilde{e} - e) \leq a(e, \tilde{e} - e) - \tilde{a}(e, \tilde{e} - e). \]

We obtain
\[ c_2 \| \tilde{e} - e \| ^2 \leq C_1 \| e \| \| \tilde{e} - e \| + C_2 \| e \| \| \tilde{e} - e \| \]
leading to
\[ \| \tilde{e} - e \| \leq \frac{C_1 + C_2}{c_2} \| e \|. \]

Altogether we obtain
\[ c_1 \| e \| ^2 \leq \| \tilde{e} \| (C_1 \| e \| + C_2 \| e - \tilde{e} \|) \leq \| \tilde{e} \| (C_1 \| e \| + C_2 \frac{C_1 + C_2}{c_2} \| e \|), \]
i.e., we have
\[ \| e \| \leq \frac{C_1 c_2 + C_2 C_1 + C_2^2}{c_1 c_2} \| \tilde{e} \|. \hspace{1cm} (2.24) \]
2.2 General solving algorithms

In this section we collect various well-known algorithms for solving variational inequalities stemming from the discretization of Signorini-type problems.

2.2.1 Relaxation methods (SOR) with subspace projection

Let \( A = (a_{ij}) \in \mathbb{R}^{N \times N} \) be a symmetric, positive definite matrix, \( b \in \mathbb{R}^N \) and \( d \in \mathbb{R}^N \). Then, the discrete variational inequality reads:

Find \( x \in K := \{ y : y_i \leq d_i, i = 1, \ldots, N \} \) such that

\[
x^T A (y - x) \geq b^T (y - x) \quad \forall y \in K.
\]  

(2.25)

The successive over-relaxation with projection is given by (see, e.g., [39]):

**Algorithm 2.1** Choose \( \omega \in (0, 2) \) (relaxation parameter).

\[
k = 0, \quad x^{(0)} \in K \text{ given}
\]

Until a stopping criterion is fulfilled, repeat:

For \( i = 1, \ldots, N \) do

\[
y_i := (1 - \omega)x_i^{(k)} + \frac{\omega}{a_{ii}} \left( b_i - \sum_{j=1,i-1}^{i+1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{N} a_{ij} x_j^{(k)} \right)
\]

\[
x_i^{(k+1)} = \min(y_i, d_i)
\]

\( k = k + 1 \)

This solver is known to be convergent (see [39]), but with a convergence rate which approaches one very fast with increasing condition number \( \kappa(A) \).

In Section 3.3 we will give several examples for the SOR method with projection for the BEM. We will also present a natural extension of this relaxation methods which is more appropriate to positive definite problems stemming from pde’s. In general the SOR method with projection has several drawbacks. First, it needs an explicit representation of the matrix, not only the action of the matrix vector multiplication. This is a severe restriction on the storage scheme for sparse matrices. Second, it is slow. There do exist some preconditioning schemes, e.g., see [39], but in general these schemes are not very efficient. Third, the SOR scheme makes no use of the origin of our problem, i.e., it does not utilize the properties of the Galerkin scheme.

2.2.2 The preconditioned Polyak algorithm

The problem given in Section 2.2.1 can be formulated as a constrained optimization problem:

\[
\min_x \frac{1}{2} x^T A x - x^T b, \quad x \in K
\]  

(2.26)
**Kuhn-Tucker optimality conditions:** Let \( x \) be arbitrary and we write \( y = Ax - b \). \( x \) solves the constrained optimization problem (2.26) if and only if the Kuhn-Tucker optimality conditions

\[
\begin{align*}
y_j & \leq 0 \text{ if } x_j = d_j \\
y_j & = 0 \text{ if } x_j < d_j
\end{align*}
\]

(2.27)

for \( j = 1, 2, \ldots, N \) are satisfied.

The Polyak algorithm (active-set strategy) constructs iterates \( x^{(k)} \leq d \) while iterating towards the correct sign conditions in (2.27).

The Polyak algorithm performs nested iterations.

**Outer iteration:** Determine

\[
I := \{ i : x_i = d_i \text{ and } y_i < 0 \}, \quad J = \{1, \ldots, N\} \setminus I
\]

for an initial \( x \leq d \). \( x_i, (i \in I) \) will be kept fixed in the inner iteration.

**Inner iteration:** \( y = Ax - b \) is split according to \( I, J \)

\[
\begin{pmatrix}
A_{I,I} & A^T_{J,I} \\
A_{J,I} & A_{J,J}
\end{pmatrix}
\begin{pmatrix}
x_I \\ x_J
\end{pmatrix}
-
\begin{pmatrix}
b_I \\ b_J
\end{pmatrix} =
\begin{pmatrix}
y_I \\ y_J
\end{pmatrix}.
\]

(2.28)

We try to force all variables \( y_J \) to be zero by solving

\[
A_{J,J}x_J = b_J - A_{J,I}x_I
\]

using the conjugate gradients algorithm (CG) with preconditioner \( B \).

If a CG step causes some variable \( x_s, s \in J \) to attain or to violate the bound, the step will be shortened to maintain feasibility. The index \( s \) is added to \( I \) and the inner iteration with be restarted with the new partition.

If the CG algorithm is complete, we have \( y_J = 0 \) and \( x_J \leq d_J \).

Then, we begin a new outer iteration with \( I \) defined by new \( x = (x_J, x_I) \).

The optimality conditions are satisfied if the index set is unchanged.

**Note 2.1** The Polyak algorithm converges since the energy is decreased for each new \( x \) (see Polyak’69 [82] and O’Leary’80 [81] for the modified scheme using a preconditioner)

**Algorithm 2.2 (Polyak)** \( I := \{1, 2, \ldots, n\}, k = 0 \)

**Outer Iteration**

\[
k = k + 1, \quad x^{(k)} = x^{(k-1)}, \quad y^{(k)} = Ax^{(k)} - b.
\]

\[
I_k = \{ i : x_i^{(k)} = d_i \text{ and } y_i^{(k)} < 0 \}
\]

**IF** \( I_k = I \) **THEN** stop **ELSE** \( I = I_k, J = \{1, 2, \ldots, n\} \setminus I \)

**Inner Iteration**

\[
\begin{pmatrix}
x^{(k)}_I \\ x^{(k)}_J
\end{pmatrix} \rightarrow \begin{pmatrix}
x^{(k)}_I \\ x^{(k)}_J
\end{pmatrix}, \quad l \rightarrow \begin{pmatrix}
l_I \\ l_J
\end{pmatrix}, \quad A \rightarrow \begin{pmatrix}
A_{II} & A^T_{JJ} \\
A_{JII} & A_{JJ}
\end{pmatrix}
\]
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Set \( q = 0 \): \[
\begin{align*}
    z^{(0)} &= x_J^{(k)} \\
    r^{(0)} &= b_J - A_J x_J^{(k)} - A_J z^{(0)}
\end{align*}
\]

Calculate search direction

if \( q = 0 \) then \( p^{(0)} = r^{(0)} \) else if \( q = 1 \) then \( p^{(1)} = Br^{(1)} \) else

\[
p^{(q)} = Br^{(q)} + \beta_q p^{(q-1)}, \quad \beta_q = \frac{(r^{(q)}, Br^{(q)})}{(r^{(q-1)}, Br^{(q-1)})}
\]

Update approximation

\[
\begin{align*}
    \alpha_q &= \min \left( \frac{(r^{(q)}, p^{(q)})}{(p^{(q)}, A_J p^{(q)})}, \min_{j \in J, j > 0} \frac{d_j - z^{(q)}_j}{p_j^{(q)}} \right) \\
    z^{(q+1)} &= z^{(q)} + \alpha_q p^{(q)} \\
    r^{(q+1)} &= r^{(q)} - \alpha_q A_J p^{(q)}
\end{align*}
\]

- IF \( r^{(q+1)} = 0 \) THEN set \( x_J^{(k)} = z^{(q+1)} \), restart outer iteration
- ELSE IF \( \{ j : z^{(q+1)}_j = d_j \} = \emptyset \) THEN \( q = q + 1 \); continue loop
- ELSE set \( x_J^{(k)} = z^{(q+1)} \), \( I = \{ i : z^{(q+1)}_i = d_i \} \).
  IF \( I = \{ 1, 2, \ldots, n \} \) THEN restart outer iteration
  ELSE restart inner iteration.

**Lemma 2.7** [81, Theorem 2] Let \( M \in \mathbb{R}^{n \times n} \) be a positive definite and symmetric preconditioner for the original matrix \( A \) with \( \kappa(MA) \ll \kappa(A) \). Then we can split \( M \) analogously to \( A \) according to the sets \( I, J \) into

\[
M = \begin{pmatrix} M_{II} & M_{JI}^T \\ M_{IJ} & M_{JJ} \end{pmatrix}
\]

and we still have \( \kappa(M_{JJ} A_{JJ}) \leq \kappa(MA) \). Therefore \( B := M_{JJ} \) is a valid preconditioner for the Polyak algorithm.

**Remark 2.2** For the computation of \( Br^{(q)} \) we don’t have to implement the matrix splitting explicitly. It is sufficient to perform \( z^{(q)} = (M(r^{(q)}, 0_I))_J \), i.e., we extend \( r^{(q)} \) by zero from the set \( J \) to \( \{ 1, \ldots, N \} \), perform the standard preconditioner and restrict the result to the set \( J \).

2.2.3 Uzawa-like algorithms for saddle-point problems

Let \( A = (a_{ij}) \in \mathbb{R}^{N \times N} \) be a symmetric, positive definite matrix, let \( B = (b_{ij}) \in \mathbb{R}^{N \times M} \), \( r \in \mathbb{R}^N \), \( l \in \mathbb{R}^M \) and \( d \in \{ \mathbb{R} \cup \infty \}^M \). Then the discrete variational inequality reads:

Find \( (\hat{x}, \hat{y}) \in \mathbb{R}^N \times K, K := \{ y : y_i \leq d_i, i = 1, \ldots, M \} \) such that

\[
\begin{align*}
\hat{x}^T A x + \hat{y}^T B^T x &= r^T x \quad \forall x \in \mathbb{R}^N \\
\hat{x}^T B (\hat{y} - \hat{y}) &\geq l^T (y - \hat{y}) \quad \forall y \in K.
\end{align*}
\]

Then the general Uzawa algorithm (see, e.g., [25]) reads:
Choose $\varrho > 0$.

**Algorithm 2.3 (Uzawa)**

$k = 0$. Initial $y^{(0)} \in K$ given.

Until a stopping criterion is fulfilled, repeat:

Solve the linear system $Ax^{(k)} = r - By^{(k)}$

Compute $y^{(k+1)} = \text{Pr}_K(y^{(k)} + \varrho(B^T x^{(k)} - l))$

$k = k + 1$

The projection onto $K$ is given component wise:

$$(\text{Pr}_K y)_i = \min(y_i, d_i)$$

**Remark 2.3** We can accelerate the solution of the linear system $Ax^{(k)} = r - By^{(k)}$ by the usual preconditioning techniques, because this system often corresponds to a well-known linear problem.
Chapter 3

BEM with Signorini contact

In this chapter we investigate the BEM problem with Signorini contact conditions for the Laplace and Lamé operators. In Section 3.1.4 we give the formulation of the boundary element methods for a general hp version and prove convergence in case of a quasi-uniform h version and a p version. In Section 3.2 we prove the properties of hierarchical error estimators based on local mesh refinements for the h version and on local subspace enrichments with bubble-functions for the h, p and hp version. In Section 3.3 we present some fast solvers for the h version BEM with Signorini contact conditions. In Section 3.4 finally we present some numerical examples.

3.1 Formulation of the BEM problem

Definition 3.1 We introduce the Sobolev spaces $H^s(\Omega)$, $H^s(\Gamma)$ and $\tilde{H}^s(\Gamma)$ as defined in the usual way. Let $\tilde{\Gamma}$ be an arbitrary closed curve or surface containing $\Gamma$. We define, as in [41, 66], the Sobolev spaces

$$
H^s(\Omega) := \{ u|_{\Omega} : u \in H^s_{\text{loc}}(\mathbb{R}^n) \}, \quad H^{-s}(\Omega) := (\tilde{H}^s(\Omega))^\prime (s \geq 0),
$$

$$
\tilde{H}^s(\Omega) := \{ u|_{\Omega} : u \in H^s_{\text{loc}}(\mathbb{R}^n), \text{supp} u \subset \Omega \},
$$

$$
H^s(\tilde{\Gamma}) := \begin{cases}
\{ v = U|_{\tilde{\Gamma}} : U \in H^{s+1/2}_{\text{loc}}(\mathbb{R}^n) \}, & s > 0, \\
L^2(\tilde{\Gamma}), & s = 0, \\
(H^{-s}(\tilde{\Gamma}))^\prime & (\text{dual space}), & s < 0.
\end{cases}
$$

Further, we define for positive $s$

$$
H^s(\Gamma) := \{ v = w|_{\Gamma} : w \in H^s(\tilde{\Gamma}) \},
$$

$$
\tilde{H}^s(\Gamma) := \{ v \in H^s(\Gamma) : \tilde{v} \in H^s(\tilde{\Gamma}) \} \quad \text{where } \tilde{v} := \begin{cases}
v & \text{on } \Gamma, \\
0 & \text{on } \tilde{\Gamma} \setminus \Gamma,
\end{cases}
$$

and for negative $s$

$$
H^s(\Gamma) := (\tilde{H}^{-s}(\Gamma))^\prime,
$$

$$
\tilde{H}^s(\Gamma) := (H^{-s}(\Gamma))^\prime.
$$
3.1.1 Laplace-BEM with Signorini contact

Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \) be a bounded domain with Lipschitz boundary \( \Gamma = \partial \Omega \) which is a disjoint union of \( \Gamma_D, \Gamma_N \) and \( \Gamma_S \neq \emptyset \). We consider the following problem.

Given \( h \in H^{-1/2}(\Gamma_N \cup \Gamma_S) \), \( g \in H^{1/2}(\Gamma_D \cup \Gamma_S) \cap C^0(\Gamma_D \cup \Gamma_S) \), find \( \hat{u} \in H^1(\Omega) \) such that

\[
\begin{align*}
\Delta \hat{u} = 0 & \quad \text{in } \Omega, \\
\hat{u} = g & \quad \text{on } \Gamma_D, \\
\frac{\partial \hat{u}}{\partial n} = h & \quad \text{on } \Gamma_N, \\
\hat{u} \leq g, \frac{\partial \hat{u}}{\partial n} \leq h, (\hat{u} - g)(\frac{\partial \hat{u}}{\partial n} - h) = 0 & \quad \text{on } \Gamma_S.
\end{align*}
\]

(3.1)

It is well known that this problem can equivalently be formulated as a variational inequality over \( \Omega \) [39] with the help of the convex subset \( \hat{K} := \{ \hat{v} \in H^1(\Omega) : \hat{v}|_{\Gamma_D} = g|_{\Gamma_D}, \hat{v}|_{\Gamma_N} \leq g|_{\Gamma_N} \} \).

Find \( \hat{u} \in \hat{K} \), such that

\[
\int_{\Omega} \nabla \hat{u} \nabla (\hat{v} - \hat{u}) \, dx \geq \int_{\Gamma_N \cup \Gamma_S} h(\hat{v} - \hat{u}) \, ds \quad \forall \hat{v} \in \hat{K}.
\]

(3.2)

**Theorem 3.1** [87] The variational inequality (3.2) has a unique solution if and only if \( \Gamma_D \neq \emptyset \) or \( \int_{\Gamma_N \cup \Gamma_S} h \, ds < 0 \).

Applying Greens’ representation formula and the jump relations to problem (3.2) we obtain a symmetric boundary integral formulation on \( \Gamma \), which is equivalent to (3.2), cf. [48].

First, we introduce the relevant boundary integral operators and review their mapping properties.

**Definition 3.2** With the fundamental solution for the Laplacian,

\[
\begin{align*}
G(x, y) &= -\frac{1}{2\pi} \log |x - y| \quad \text{if } n = 2, \\
G(x, y) &= \frac{\Gamma(n-2)}{4\pi^{n/2}} |x - y|^{2-n} \quad \text{if } n \geq 3,
\end{align*}
\]

we define the single layer potential \( V \), the double layer potential \( K \), its formal adjoint \( K' \), and the hypersingular integral operator \( W \) for \( z \in \Gamma \), \( \phi \in C^\infty(\Gamma) \) as follows:

\[
\begin{align*}
V\phi(z) &:= 2 \int_{\Gamma} G(z, x) \cdot \phi(x) \, ds_x, \\
K\phi(z) &:= 2 \int_{\Gamma} \frac{\partial}{\partial n_z} G(z, x) \cdot \phi(x) \, ds_x, \\
K'\phi(z) &:= 2 \int_{\Gamma} \frac{\partial}{\partial n_z} G(z, x) \cdot \phi(x) \, ds_x, \\
W\phi(z) &:= -\frac{\partial}{\partial n_z} K\phi(z).
\end{align*}
\]

For a distribution \( w \) on \( \Gamma \) we define (if possible) \( Vw, Kw, \) etc. by approximating \( w \) with smooth functions.

**Lemma 3.1** Let \( \Gamma \) be a Lipschitz boundary and \( |\sigma| < 1/2 \). From [27, Theorem 1],[28] we know that

\[
\begin{align*}
V : H^{-1/2+\sigma}(\Gamma) \to H^{1/2+\sigma}(\Gamma), & \quad K : H^{1/2+\sigma}(\Gamma) \to H^{1/2+\sigma}(\Gamma), \\
K' : H^{-1/2+\sigma}(\Gamma) \to H^{-1/2+\sigma}(\Gamma), & \quad W : H^{1/2+\sigma}(\Gamma) \to H^{-1/2+\sigma}(\Gamma)
\end{align*}
\]

are linear and continuous. If \( \text{diam}\(\Omega\) is small enough, \( V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \) and \( W : H^{1/2}(\Gamma)/\mathbb{R} \to H^{-1/2}(\Gamma) \) are strongly positive definite.
Then, an equivalent formulation to (3.1) is given by problem (L):

**Definition 3.3** Let $K^\Gamma := \{ v \in H^{1/2}(\Gamma) : v|_{\Gamma_D} = g|_{\Gamma_D}, v|_{\Gamma_S} \leq g|_{\Gamma_S} \}$.

The problem (L) reads:

Find $(u, \varphi) \in K^\Gamma \times H^{-1/2}(\Gamma)$ such that

\[
\begin{align*}
\langle u, W(v-u) \rangle + \langle \varphi, (I+K)(v-u) \rangle & \geq 2l(v-u) \\
\langle \psi, V\varphi \rangle - \langle \psi, (I+K)u \rangle & = 0 \quad \forall (v, \psi) \in K^\Gamma \times H^{-1/2}(\Gamma)
\end{align*}
\]  

(3.3)

using

\[
l(v) = \int_{\Gamma_N \cup \Gamma_S} hv \, ds
\]

and the integral operators defined above. Equivalently, we can write the system (3.3) in form of a coercive and non-symmetric bilinear form, i.e., let

\[
B(u, \varphi; v, \psi) := \langle Wu, v \rangle + \langle (I+K)^T \varphi, v \rangle + \langle V\varphi, \psi \rangle - \langle (I+K)u, \psi \rangle
\]

\[
\mathcal{L}(v, \psi) := 2 \int_{\Gamma_N \cup \Gamma_S} hv \, ds.
\]

Then the variational inequality reads:

Find $(u, \varphi) \in K^\Gamma \times H^{-1/2}(\Gamma)$ such that

\[
B(u, \varphi; v-u, \psi) \geq \mathcal{L}(v-u, \psi) \quad \forall (v, \psi) \in K^\Gamma \times H^{-1/2}(\Gamma).
\]  

(3.4)

**Definition 3.4** Using the symmetric Steklov-Poincaré operator for the interior problem

\[
S := \frac{1}{2}(W + (K' + I)V^{-1}(K + I)) : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)
\]  

(3.5)

which is positive definite on $H^{1/2}(\Gamma)/\mathbb{R}$, we can write (3.4) equivalently as problem (S):

Find $u \in K^\Gamma$ such that

\[
\langle Su, v-u \rangle \geq l(v-u) \quad \forall v \in K^\Gamma.
\]  

(3.6)

Existence and uniqueness of the solution of problems (S) and (L), respectively, have been shown by Houde Han [50]; for the corresponding elasticity problem see Gwinner&Stephan [48].

### 3.1.2 Numerical approximation

Let $\omega_h$ and $\gamma_h$ be two not necessarily identical regular partitions of $\Gamma$, such that all corners of $\Gamma$ and all “end points” $\Gamma_S \cap \Gamma_N, \Gamma_N \cap \Gamma_D, \Gamma_D \cap \Gamma_S$ are nodes of the partitions $\omega_h$ and $\gamma_h$. Often we use $\omega_h = \gamma_h$, but this is only necessary when we are approximating the boundary (see, e.g., example 3.3 for the Lamé equation dealing with the Hertz contact problem).

Let $\vec{p} = (p_e)_{e \in \omega_h}$, or $\vec{q} = (q_e)_{e \in \gamma_h}$ be degree vectors which associate each element of $\omega_h$ or $\gamma_h$ with a polynomial degree $p_e \geq 1$ or $q_e \geq 0$.

On the interval $[-1, 1]$ we choose $N + 1$ Gauss-Lobatto points, i.e., the points $\xi_j^{N+1}, 0 \leq j \leq N$, that are the zeros of $(1 - \xi^2)L'_N(\xi)$, where $L_N$ denotes the Legendre
polynomial of degree $N$. For these points it is known (cf. [10, Prop. 2.2, (2.3)]) that there exist positive weight factors $\phi_j^{N+1} := \frac{1}{N(N+1)E_N(\xi_j^{N+1})}$ such that
\[
\forall \phi \in P_{2N-1}([-1, 1]) : \sum_{j=0}^{N} \phi(\xi_j) \phi_j^{N+1} = \int_{-1}^{1} \phi(\xi) \, d\xi. \tag{3.7}
\]

By an affine transformation we define the set of Gauss-Lobatto points $G_{e, hp}$ on each element $e$ of the partition $\omega_h$ of $\Gamma$, corresponding to the polynomial degree $p_e$, i.e., we use the Gauss-Lobatto formula with $p_e + 1$ nodes (see [65]) and we set
\[
G_{hp} := \bigcup_{e \in \omega_h} G_{e, hp}.
\]

On the partition $\omega_h$ of $\Gamma$ we define $\sigma_{hp}$ as the space of continuous polynomials with $\sigma_{hp}|e \subseteq P_{p_e}(e), \forall e \in \omega_h$. A suitable basis of $\sigma_{hp}$ is given by the Lagrange interpolation polynomials on the set of Gauss-Lobatto points of each element, see Figure 3.1, because later we have to evaluate the values of the solution on the set $G_{hp}$.

On the partition $\gamma_h$ of $\Gamma$ we introduce $\tau_{hp}$ as the space of piecewise polynomials with $\tau_{hp}|e \subseteq P_{q_e}(e), \forall e \in \gamma_h$; a suitable basis is given by the Legendre polynomials.

Thus, our test and trial spaces are defined as
\[
\sigma_{hp} := \{v_{hp} \in C^0(\Gamma; \mathbb{R}) : v_{hp}|e \in P_{p_e}(e), \forall e \in \omega_h\}, \tag{3.8}
\]
\[
\tau_{hp} := \{\psi_{hp} \in L^2(\Gamma; \mathbb{R}) : \psi_{hp}|e \in P_{q_e}(e), \forall e \in \gamma_h\}. \tag{3.9}
\]

![Figure 3.1: Lagrange polynomials, $N = 5$](image)

Now, we choose
\[
K_{hp}^{\Gamma} := \{v \in \sigma_{hp} | \forall x \in G_{hp} \cap \Gamma_S : v(x) \leq g(x) \text{ and } \forall x \in G_{hp} \cap \Gamma_D : v(x) = g(x)\}. \tag{3.10}
\]

which is a convex, closed subset of $\sigma_{hp}$. Note that $K_{hp}^{\Gamma} \subseteq K^\Gamma$ for $p \geq 2$ (as Figure 3.1 illustrates) or for non-concave contact function $g$.

Based on the local set of Gauss-Lobatto points $G_{e, hp}$ we define the interpolation operator $i_{e, p_e} : C^0(\bar{e}) \to P_{p_e}(e)$ by
\[
(i_{e, p_e} \psi)(x) = \psi(x) \quad \forall x \in G_{e, hp}, \forall \psi \in C^0(\bar{e})
\]
and the global interpolation operator \( i_{\omega_h, p} : C^0(\Gamma) \rightarrow \sigma_{hp} \) by
\[
i_{\omega_h, p} \psi := \sum_{e \in \omega_h} \chi_e i_{e, p} \psi |_e \quad \forall \psi \in C^0(\Gamma),
\] (3.11)
where \( \chi_e \) is the characteristic function of \( e \in \omega_h \).

**Definition 3.5** The discrete problem \((L_{hp})\) of the general \( hp \) version reads:
Find \((u_{hp}, \varphi_{hp}) \in K_{hp}^\Gamma \times \tau_{hp}\) such that
\[
B(u_{hp}, \varphi_{hp}; v - u_{hp}, \psi) \geq 2l(v - u_{hp}) \quad \forall (v, \psi) \in K_{hp}^\Gamma \times \tau_{hp}.
\] (3.12)
Via the canonical imbeddings \( j_{hp} : \sigma_{hp} \hookrightarrow H^{1/2}(\Gamma) \) and \( k_{hp} : \tau_{hp} \hookrightarrow H^{-1/2}(\Gamma) \) and their duals \( j_{hp}^* \) and \( k_{hp}^* \) the discrete Steklov-Poincaré operator \( S_{hp} : \sigma_{hp} \rightarrow \sigma_{hp}^* \)
\[
S_{hp} := \frac{1}{2} (j_{hp}^* W_{j_{hp}} + j_{hp}(I + K^t) k_{hp}(k_{hp}^* V k_{hp})^{-1} k_{hp}^* (I + K) j_{hp})
\] (3.13)
is well defined (see [19]).

**Definition 3.6** With (3.13) the discrete problem \((S_{hp})\) of the general \( hp \) version (3.6) reads:
Find \( u_{hp} \in K_{hp}^\Gamma \) such that
\[
\langle S_{hp} u_{hp}, v_{hp} - u_{hp} \rangle \geq l(v_{hp} - u_{hp}) \quad \forall v_{hp} \in K_{hp}^\Gamma.
\] (3.14)

**h version** Let \( p \in \mathbb{N}, p \geq 1 \), arbitrary but fixed. Let the degree vectors \( \vec{p} = (p_e)_{e \in \omega_h} \) and \( \vec{q} = (q_e)_{e \in \gamma_h} \) be fixed with \( p_e = p, \forall e \in \omega_h \) and \( q_e = p - 1, \forall e \in \gamma_h \), and let \( \omega_h, \gamma_h \) be quasi-uniform partitions. In this case we write \( \sigma_h, \tau_h, G_h, K_h^\Gamma \) instead of \( \sigma_{hp}, \tau_{hp}, \) etc. In case of \( p = 1 \) the definition of \( \sigma_h, \tau_h, G_h, K_h^\Gamma \) simplifies to
\[
\sigma_h := \{v_h \in C^0(\Gamma; \mathbb{R}) ; v_h \text{ p.w. linear on } \omega_h\},
\] (3.15)
\[
\tau_h := \{\psi_h \in L^2(\Gamma; \mathbb{R}) ; \psi_h \text{ p.w. constant on } \gamma_h\},
\] (3.16)
\[
G_h := \{x \in \Gamma : x \text{ is node of the partition } \omega_h\},
\] (3.17)
\[
K_h^\Gamma := \{v \in \sigma_h \mid \forall x \in G_h \cap \Gamma_S : v(x) \leq g(x), \forall x \in G_h \cap \Gamma_D : v(x) = g(x)\}. \] (3.18)
Let \( n_h = \dim \sigma_h, m_h = \dim \tau_h\). Let \( \sigma_h = \text{span}\{\phi_i, i = 1, \ldots, n_h\} \) with \( \phi_i \) being hat-functions with \( \phi_i(p_j^{(h)}) = \delta_{ij} \) for the corresponding nodes \( p_j^{(h)} \in G_h\).

**Definition 3.7** The discrete problem \((L_h)\) of the \( h \) version now reads:
Find \((u_h, \varphi_h) \in K_h^\Gamma \times \tau_h\) such that
\[
B(u_h, \varphi_h; v - u_h, \psi) \geq 2l(v - u_h) \quad \forall (v, \psi) \in K_h^\Gamma \times \tau_h.
\] (3.19)

**p version** Let \( \omega_h, \gamma_h \) be fixed and let the degree vectors \( \vec{p} = (p_e)_{e \in \omega_h} \) and \( \vec{q} = (q_e)_{e \in \gamma_h} \) be independent of \( e \), i.e., we have \( p_e = p \), \( \forall e \in \omega_h \) and \( q_e = p - 1 \), \( \forall e \in \gamma_h \). In this case we write \( \sigma_p, \tau_p \) instead of \( \sigma_{hp}, \tau_{hp}, \) etc.
There holds the following convergence result for the Galerkin solution of \((L_{hp})\) in the energy norm without any regularity assumptions.
CHAPTER 3. BEM WITH SIGNORINI CONTACT

**Theorem 3.2** Let \((\omega_h, \gamma_h)_{h \in I}\) be a family of quasi-uniform meshes, such that \(h := \max\{|e|, e \in \omega_h\} = \gamma_h\), where \(I \subseteq (0, \infty)\) with \(0 \in I\). Let \(\tilde{p} = (p_e)_{e \in \omega_h}\), such that \(p_e = p\) for all \(e \in \omega_h\), and let \(q_e = (q_e)_{e \in \gamma_h}\), such that \(q_e = p - 1\) for all \(e \in \gamma_h\).

Let the solutions \((u, \varphi)\) of (3.4) and \((u_{hp}, \varphi_{hp})\) of (3.12) exist uniquely. Suppose that for the polygonal domain \(\Omega\), there are only a finite number of ”end points” \(\Gamma_S \cap \Gamma_D, \Gamma_D \cap \Gamma_N, \Gamma_N \cap \Gamma_S\). Then there hold

\[
\lim_{p \to \infty} \|u - u_{hp}, \varphi - \varphi_{hp}\|_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)} = 0, \text{ if } h \text{ fixed}
\]

and

\[
\lim_{h \to 0} \|u - u_{hp}, \varphi - \varphi_{hp}\|_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)} = 0, \text{ if } p \text{ fixed}.
\]

**Remark 3.1** The proof is shown explicitly for \(n = 2\). Since the basic approximation arguments are still valid for \(n = 3\), at least in the case of parallelogram meshes, the argumentation also holds for \(n = 3\) and is omitted for brevity.

**Proof.** Since the bilinear form \(B(\cdot, \cdot)\) is positive definite we can apply [39, Theorem I.5.2] (or [47, Theorem 4.1]) and therefore we require the following hypotheses:

**H1** If \(\{v_{hp}\}_{hp}\) converges weakly in \(H^{1/2}(\Gamma)\) to \(v\) (for \(h \to 0\) or \(p \to \infty\)), where \(v_{hp} \in K_{hp}\), then \(v \in K\).

**H2** There exist a subset \(M \subset H^{1/2}(\Gamma)\) such that \(\tilde{M} = K^\Gamma\) and mappings \(g_{hp} : M \to \sigma_{hp}\) such that, for each \(w \in M\), \(g_{hp}w\) converges strongly to \(w\) (as \(p \to \infty\), or \(h \to 0\)) and \(g_{hp}w \in K_{hp}\) for all \(p < \infty, h > 0\).

The analogous hypotheses for the approximation of \(\psi \in H^{-1/2}(\Gamma)\) by \(\psi_{hp} \in \tau_{hp} \subset H^{-1/2}(\Gamma)\) are trivially satisfied by well-known density and approximation properties.

**Verification of H1.** Consider \(\phi \in C^0(\Gamma)\) with \(\phi|\Gamma_S \geq 0\). For \(e \in \omega_h\) we approximate \(\phi\) by a combination of Bernstein polynomials \(B_{e, p_e}\) on the elements \(e \in \omega_h\), i.e., with the local mapping \(F_e : \left\{\begin{array}{ccc} [0, 1] & \to & e \\ t & \to & x(t) \end{array}\right.\) we define

\[
\phi_{e, p}(t) := B_{e, p_e}\phi(F_e(t)) := \sum_{k=0}^{p_e} \binom{p_e}{k} t^k (1 - t)^{p_e - k}\phi(F_e(\frac{k}{p_e})).
\]

With the characteristic function \(\chi_e\) of \(e \in \omega_h\) we set

\[
\phi_{hp} := B_{\omega_h, p}\phi := \sum_{e \in \omega_h} \chi_e(x)\phi_{e, p}(F_e^{-1}(x)),
\]

yielding \(\phi_{hp} \in \sigma_{hp}\). Since the Bernstein operators are monotone we have \(\phi_{hp}|\Gamma_S \geq 0\).

With [99, Theorem 2.3] we have

\[
\lim_{p \to \infty} \|\phi - \phi_{hp}\|_{L^\infty(\Gamma)} = 0, \text{ if } h \text{ fixed}. \tag{3.20}
\]

By the uniform continuity of \(\phi \in C^0(\Gamma)\) we have that

\[
\lim_{h \to 0} \|\phi - \phi_{hp}\|_{L^\infty(\Gamma)} = 0, \text{ if } p \text{ fixed}. \tag{3.21}
\]
For the contact function \( g \in H^{1/2}(\Gamma_D \cup \Gamma_S) \cap C^0(\Gamma_D \cup \Gamma_S) \) we define the interpolate \( g_{hp} := i_{\omega_h|_{\Gamma_D \cup \Gamma_S}}’g \). By [10, Theorem 4.2] we know
\[
\lim_{p \to \infty} \|g - g_{hp}\|_{L^2(\Gamma_D \cup \Gamma_S)} = 0, \text{ if } h \text{ fixed}
\]
and
\[
\lim_{h \to 0} \|g - g_{hp}\|_{L^2(\Gamma_D \cup \Gamma_S)} = 0, \text{ if } p \text{ fixed}.
\]
Since the embedding \( H^{1/2}(\Gamma) \subset L^1(\Gamma_S) \) is weakly continuous, \( v_{hp} \in K^\Gamma_{hp} \) converges weakly to \( v \) in \( L^1(\Gamma_S) \) and \( \|v_{hp}\|_{L^1(\Gamma_S)} \) is bounded. Therefore by the estimate
\[
\left| \int_{\Gamma_S} ((v_{hp} - g_{hp})\phi_{h,p-1} - (v - g)\phi) \, ds \right| \leq \|v_{hp} - g_{hp}\|_{L^1(\Gamma_S)}\|\phi_{h,p-1} - \phi\|_{L^\infty(\Gamma_S)} + \int_{\Gamma_S} (v_{hp} - g_{hp} - (v - g))\phi \, ds,
\]
and using (3.20), (3.21) and \( g \in L^1(\Gamma_S) = (L^\infty(\Gamma_S))^* \), we obtain that
\[
\lim_{p \to \infty} \int_{\Gamma_S} (v_{hp} - g_{hp})\phi_{h,p-1} \, ds = \int_{\Gamma_S} (v - g)\phi \, ds, \text{ if } h \text{ fixed}
\]
and
\[
\lim_{h \to 0} \int_{\Gamma_S} (v_{hp} - g_{hp})\phi_{h,p-1} \, ds = \int_{\Gamma_S} (v - g)\phi \, ds, \text{ if } p \text{ fixed}.
\]
The function \( (v_{hp} - g_{hp})\phi_{h,p-1}|_e, e \in \omega_h|_{\Gamma_S} \), is a polynomial of degree \( 2p_e - 1 \). Therefore, the numerical quadrature using \( p_e + 1 \) Gauss-Lobatto nodes is exact, i.e., with (3.7) and the definition of \( g_{hp} \) we get for all \( e \in \omega_h|_{\Gamma_S} \)
\[
\int_e (v_{hp} - g_{hp})\phi_{h,p-1} \, ds = \frac{|e|}{2} \sum_{j=0}^{p_e} [(v_{hp} - g_{hp})\phi_{e,p_e-1}](F_e(\epsilon^2_j + 1)\phi^j_{p_e+1}).
\]
Due to \( \phi_{e,p_e-1} \geq 0 \), \( (v_{hp} - g_{hp})(x) \leq 0 \) for all \( x \in G_{hp} \cap \Gamma_S \) and the positivity of the weights \( \phi^j_{p_e+1} \) of the Gauss-Lobatto quadrature formula, we have
\[
\int_e (v_{hp} - g_{hp})\phi_{e,p_e-1} \, ds \leq 0 \quad \forall e \in \omega_h|_{\Gamma_S}.
\]
Therefore it follows for all \( \phi \in C^0(\Gamma) \) with \( \phi|_{\Gamma_S} \geq 0 \) that
\[
\int_{\Gamma_S} (v - g)\phi \, ds \leq 0.
\]
Hence \( v \leq g \) almost everywhere on \( \Gamma_S \), i.e., \( v \in K^\Gamma \).

**Verification of H2.** Due to [48, Lemma 3.3] we have the density relation
\[
K^\Gamma \cap C^\infty(\Gamma) = K^\Gamma.
\]
Therefore we can take \( M = K^\Gamma \cap C^\infty(\Gamma) \).

We define \( g_{hp} : M \to \sigma_{hp} \) by
\[
g_{hp}v := i_{\omega_h,v}.
\]
As shown in [10, Theorem 4.7] there exists a constant $C$ independent of $v$ and $\tilde{p}, \omega_h$ such that
\[
\|v - \vartheta_h v\|_{H^{1/2}(\Gamma)} \leq C h^{1/2} p^{-1/2} \|v\|_{H^1(\Gamma)} \quad \forall v \in H^1(\Gamma).
\]
Therefore, the sequence $\vartheta_h v$ converges strongly to $v \in M$, if $p$ tends to infinity and $h$ is fixed, or if $h$ tends to zero and $p$ is fixed.

Due to $v \in K^1$, i.e., $v(x) \leq g(x)$ for all $x \in \Gamma_S$, and the interpolation property of $i_{\omega_h} \vartheta_h^v$, we have $\vartheta_h v(x) \leq g(x)$ for all $x \in G_{hp} \cap \Gamma_S$ and $\vartheta_h v(x) = g(x)$ for all $x \in G_{hp} \cap \Gamma_D$, i.e., $\vartheta_h v \in K_{hp}^1$.

\textbf{Remark 3.2} For a proof in case of the $h$ version see Gwinner/Stephan [48], where Newton-Cotes formulas were used as numerical quadrature rules; here Gauss-Lobatto quadrature is used instead.

Alternatively, we can state problems (3.19) and (3.14), respectively, in matrix form.

Let $\{\phi_i : i = 1, \ldots, n_{hp}\}$, $n_{hp} := \dim \sigma_{hp}$, be a basis of $\sigma_{hp}$ and let $\{\beta_j : j = 1, \ldots, m_{hp}\}$, $m_{hp} := \dim \tau_{hp}$, be a basis of $\tau_{hp}$.

Then we can define the Galerkin matrices by $W_N = (W_{ij})_{ij} \in \mathbb{R}^{n_{hp} \times m_{hp}}$, $W_{ij} = (\phi_i, W \phi_j)$, etc., and $\bar{I} := (I_{ij} = (I(\phi_i))_i)$.

\textbf{Discrete formulation I:}

Find $\tilde{u} := (u_i) \in K_{n_{hp}}$, $\varphi := (\varphi_j) \in \mathbb{R}^{m_{hp}}$, such that
\[
\begin{align*}
\sum_{j} u_j W_N (\bar{v} - \bar{u}) + \varphi^T (I + K)_N (\bar{v} - \bar{u}) &\geq 2 \bar{u}(\bar{v} - \bar{u}) \\
\varphi^T V_N \varphi - \varphi^T (I + K)_N \bar{u} &\geq 0
\end{align*}
\]
(3.28)
\[
\forall \bar{v} = (v_1, \ldots, v_{n_{hp}}) \in K_{n_{hp}}, \varphi \in \mathbb{R}^{m_{hp}} \text{ with }
\]
\[
K_{n_{hp}} := \{\bar{v} = (v_1, \ldots, v_{n_h}) : v_i = g(x_i), x_i \in G_{hp} \cap \Gamma_D; v_i \leq g(x_i), x_i \in G_{hp} \cap \Gamma_S\}.
\]

Inserting
\[
\bar{\varphi} = V_N^{-1} (I + K)_N \bar{u}
\]
in (3.28) yields the Schur complement
\[
\tilde{u}^T \left[ W_N + (I + K)_N V_N^{-1} (I + K)_N \right] (\bar{v} - \bar{u}) \geq 2 \bar{u}(\bar{v} - \bar{u})
\]
which leads to the matrix form of (3.14).

\textbf{Discrete formulation II:}

Find $\tilde{u} = (u_1, \ldots, u_{n_{hp}}) \in K_{n_{hp}}$, such that
\[
\tilde{u}^T S_N (\bar{v} - \bar{u}) \geq \bar{I}(\bar{v} - \bar{u})
\]
(3.29)
for all $\bar{v} \in K_{n_{hp}} \subset \mathbb{R}^{n_{hp}}$.

\textbf{Lemma 3.2} Let $J(v) := \frac{1}{2} (Sv, v) - l(v)$, $\forall v \in H^{1/2}(\Gamma)/\mathbb{R}$ and let $J_h(v_h) := \frac{1}{2} (S_h v_h, v_h) - l(v_h)$, $\forall v_h \in \sigma_h$. Let $K_1^{h} \subseteq K$. Let $u$ be the solution of (3.6) and let $u_h$ be the solution of (3.19). Then there holds
\[
\|u - u_h\|_{H^{1/2}(\Gamma)} \leq C (J_h(u_h) - J(u)) + C (e_h(u) + \|u - u_h\|_{H^{1/2}(\Gamma)})^2
\]
(3.30)
for a constant $C$ independent of $h$, where $e_h(u) := \text{dist}_{H^{-1/2}(\Gamma)} (V^{-1}(I + K)u, \tau_h)$. 

Proof. Let \( A = W, B = I + K', C = I + K, D = V, X = H^{1/2}(\Gamma)/\mathbb{R} \) and \( Y = H^{-1/2}(\Gamma) \), then we have from Lemma 2.4 that

\[
\|u - u_h\|_2^2 \leq 2(J_h(u_h) - J(u)) + c(e_h(u) + \|u - u_h\|_{H^{1/2}(\Gamma)/\mathbb{R}})^2
\]

With \( \| \cdot \|_{H^{1/2}(\Gamma)/\mathbb{R}} \sim \| \cdot \|_S \) the assertion follows. \( \Box \)

### 3.1.3 Lamé-BEM with Signorini contact

We can treat the Lamé operator similarly to the Laplace operator:

Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \) be a bounded domain with Lipschitz boundary \( \Gamma = \partial \Omega \) which is a disjoint union of \( \Gamma_D, \Gamma_N \) and \( \Gamma_S \neq \emptyset \). We consider the following problem:

Given \( h \in [H^{-1/2}(\Gamma_N \cup \Gamma_S)]^n, g \in [H^{1/2}(\Gamma_D \cup \Gamma_S)]^n \cap [C_0(\Gamma_D \cup \Gamma_S)]^n \), find \( \hat{u} \in [H^1(\Omega)]^n \) such that

\[
\Delta^* \hat{u} := \mu \Delta \hat{u} + (\lambda + \mu) \text{grad} \text{div} u = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n,
\]

\[
\hat{u} = g \quad \text{on} \quad \Gamma_D, \quad \hat{\phi} := T(\hat{u}) = h \quad \text{on} \quad \Gamma_N,
\]

\[
\hat{u}_n \leq g_n \quad \text{on} \quad \Gamma_S, \quad \hat{\phi}_n \leq h_n \quad \text{on} \quad \Gamma_S,
\]

\[
0 = (\hat{u}_n - g_n)(\hat{\phi}_n - h_n), \quad \hat{\phi}_n = h_n \quad \text{on} \quad \Gamma_S,
\]

where we have \( \hat{u}_n = \hat{u} \cdot \mathbf{n}, \hat{u}_t = \hat{u} - (\hat{u} \cdot \mathbf{n}) \mathbf{n} \) (tangential components) etc. on \( \Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_S \). \( \lambda, \mu \) are the Lamé-constants.

The traction operator \( T \) is defined (component wise) for \( n \geq 2 \) by

\[
(Tu)_i = \lambda n_i \text{div} u + \mu \frac{\partial}{\partial n} u_i + \mu \frac{\partial u}{\partial x_i} \mathbf{n}.
\]

For \( n = 2 \) it has the following equivalent representation

\[
T(u) = \lambda (\text{div} u) \mathbf{n} + 2\mu \frac{\partial u}{\partial n} + \mu \mathbf{n} \times \text{rot} u.
\]

It is well known that the problem (3.31) can equivalently be formulated as a variational inequality over \( \Omega \), e.g., see [33], [62].

Find \( \hat{u} \in K := \{ v \in [H^1(\Omega)]^n : v|_{\Gamma_D} = g|_{\Gamma_D}, v \cdot \mathbf{n}|_{\Gamma_S} \leq g_n \} \), with

\[
\beta(\hat{u}, v - \hat{u}) \geq l(v - \hat{u}) \quad \forall v \in K,
\]

using the bilinear form \( \beta(v, w) \), representing the strain energy,

\[
\beta(v, w) := \mu \sum_{j=1}^n \int_{\Omega} \nabla v_j \nabla w_j + (\lambda + \mu) \int_{\Omega} \text{div} v \text{div} w
\]

and the linear functional \( l(v) \), representing the work done by the exterior forces,

\[
l(v) := \int_{\Gamma_N \cup \Gamma_S} h \cdot v ds.
\]

Let \( \mathcal{R} \) be the space of rigid body movements, e.g., for \( n = 2 \) we have \( \mathcal{R} = \{ r \in [H^1(\Omega)]^2 : r = (\omega_1 + \omega_3 x_2, \omega_2 - \omega_3 x_1) ; \omega_1, \omega_2, \omega_3 \in \mathbb{R} \} \). Then there holds:
Theorem 3.3 [37]/[62, Theorem 6.1] If
\[ l(r) < 0, \quad \forall r \in \mathcal{R} \cap K \setminus \{0\} \]
with \( \mathcal{R} \) be the space of rigid body movements, or \( \Gamma_D \neq \emptyset \), then problem (3.34) is uniquely solvable.

Applying Green’s representation formula and the jump relations to problem (3.34) we obtain a boundary integral formulation which is equivalent to (3.34), see [48, Section 3].

Definition 3.8 The fundamental solution of the Lamé operator \( \Delta^* \) in \( \mathbb{R}^n \) is given by
\[ S(x, y) = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)} \left\{ G(x, y) I + \frac{\lambda + \mu}{\omega_n(\lambda + 3\mu)} (x - y)(x - y)^t \right\} \]
where \( G(x, y) \) is the fundamental solution of the Laplacian (see Def. 3.2) and
\[ \omega_2 = 2\pi, \quad \omega_3 = 4\pi, \quad \omega_n = \frac{\Gamma(n-2)}{4\pi n^2/2} \quad (n \geq 3). \]

Then the boundary integral operators are given by
\[ V u(x) := 2 \int_{\Gamma} S(x, y) u(y) \, ds_y, \quad K u(x) := 2 \int_{\Gamma} (T_y S(x, y))^t u(y) \, ds_y, \]
\[ K' u(x) := 2 \int_{\Gamma} T_x S(x, y) u(y) \, ds_y, \quad W u(x) := -T_x K u(x) \, ds_y. \]

Definition 3.9 Find \((u, \varphi) \in K^\Gamma \times [H^{-1/2}(\Gamma)]^n\):
\[ \langle u, W(v-u) \rangle + \langle \varphi, (I+K)(v-u) \rangle \geq 2 \int_{\Gamma_N \cup \Gamma_S} h(v-u) \, ds, \]
\[ \langle \psi, V \varphi \rangle - \langle \psi, (I+K)u \rangle = 0 \]
for all \((v, \psi) \in K^\Gamma \times [H^{-1/2}(\Gamma)]^n\).

The convex cone \( K^\Gamma \) is given by
\[ K^\Gamma := \{ v \in [H^{1/2}(\Gamma)]^n : v|_{\Gamma_D} = g|_{\Gamma_D}, v_n|_{\Gamma_S} \leq g_n|_{\Gamma_S} \}. \]

Equivalently, we can write the system (3.37) in form of a coercive and non-symmetric bilinear form, i.e., let
\[ B(u, \varphi; v, \psi) := \langle W u, v \rangle + \langle (I+K)^t \varphi, v \rangle + \langle V \varphi, \psi \rangle - \langle (I+K)u, \psi \rangle, \]
\[ L(v, \psi) := 2 \int_{\Gamma_N \cup \Gamma_S} h v \, ds. \]

Then, the variational inequality reads:
Find \((u, \varphi) \in K^\Gamma \times [H^{-1/2}(\Gamma)]^n\) such that
\[ B(u, \varphi; v-u, \psi) \geq L(v-u, \psi) \quad \forall (v, \psi) \in K^\Gamma \times H^{-1/2}(\Gamma). \]

The Steklov-Poincaré operator for this interior problem is defined by
\[ S := \frac{1}{2} \left( W + (K' + I)W^{-1}(K + I) \right) : [H^{1/2}(\Gamma)]^n \rightarrow [H^{-1/2}(\Gamma)]^n. \]

Theorem 3.4 (Gwinner, Stephan’93 [48]) Inequalities (3.37) are uniquely solvable and equivalent to inequality (3.34).
3.1.4 Numerical approximation

The discretization of (3.37) is done analogously to problem (L) for the Laplace operator. Let \( \omega_h, \gamma_h \) and \( G_{hp} \) defined as in Section 3.1.2. Then we can define our test and trial spaces as

\[
\sigma_{hp} := \{v_{hp} \in C^0(\Gamma; \mathbb{R}^n) : e \in \mathcal{P}_{p_e}(e)^n, \forall e \in \omega_h\},
\]

\[
\tau_{hp} := \{\psi_{hp} \in L^2(\Gamma; \mathbb{R}^n) : e \in \mathcal{P}_{q_e}(e)^n, \forall e \in \gamma_h\}.
\]

And the convex, closed subset \( K^\Gamma_{hp} \) of \( \sigma_{hp} \) is given by

\[
K^\Gamma_{hp} := \{v \in \sigma_{hp} \mid \forall x \in G_{hp} \cap \Gamma_S : \mathbf{n} \cdot v(x) \leq g_n(x) \text{ and } \forall x \in G_{hp} \cap \Gamma_D : v(x) = g(x)\}.
\]

**Definition 3.10** The discrete version of (3.38) of the general \( hp \) version reads:

Find \( (u_{hp}, \varphi_{hp}) \in K^\Gamma_{hp} \times \tau_{hp} \) such that

\[
B(u_{hp}, \varphi_{hp}; v - u_{hp}, \psi) \geq 2l(v - u_{hp}) \quad \forall (v, \psi) \in K^\Gamma_{hp} \times \tau_{hp}.
\]

Then there holds

**Theorem 3.5** Let \( n = 2 \). Let \( (\omega_h, \gamma_h)_{h \in I} \) be a family of quasi-uniform meshes, such that \( h := \max\{|e|, e \in \omega_h \text{ or } e \in \gamma_h\} \), where \( I \subseteq (0, \infty) \) with \( 0 \in I \). Let \( \bar{p} = (p_e)_{e \in \omega_h} \), such that \( p_e = p \) for all \( e \in \omega_h \), and let \( \bar{q} = (q_e)_{e \in \gamma_h} \), such that \( q_e = p - 1 \) for all \( e \in \gamma_h \). Let the solutions \( (u, \varphi) \) to (3.38) and \( (u_{hp}, \varphi_{hp}) \) to (3.43) exist uniquely. Suppose that for the polygonal domain \( \Omega \), there are only a finite number of ”end points” \( \Gamma_S \cap \Gamma_D, \Gamma_D \cap \Gamma_N, \Gamma_N \cap \Gamma_S \). Then there hold

\[
\lim_{p \to \infty} ||(u - u_{hp}, \varphi - \varphi_{hp})||_{H^{1/2}(\Gamma)^n \times [H^{-1/2}(\Gamma)]^n} = 0, \text{ if } h \text{ fixed}
\]

and

\[
\lim_{h \to 0} ||(u - u_{hp}, \varphi - \varphi_{hp})||_{H^{1/2}(\Gamma)^n \times [H^{-1/2}(\Gamma)]^n} = 0, \text{ if } p \text{ fixed}.
\]

**Proof.** Analogously to the proof of Theorem 3.2. \( \square \)
3.2 A posteriori error estimate for BEM with Signorini contact

In this section we present a posteriori error estimators for the BEM in two dimensions. The results can be easily generalized to higher dimensions.

Let \( k_u := \frac{h}{2} \) and \( k_V := \frac{h'}{2} \). For all \( (u', \varphi) \in H^1(\Gamma)/\mathbb{R} \times H^{-1}(\Gamma) \) we define the norm \( \| (u, \varphi) \|_{\mathcal{H}} = (\|u\|_{H^1(\Gamma)/\mathbb{R}}^2 + \|\varphi\|_{H^{-1}(\Gamma)}^2)^{1/2} \), which is equivalent to \( \| (u, \varphi) \|_{\mathcal{H}} = (\|u\|_{H_{h/2}^1(\Gamma)/\mathbb{R}}^2 + \|\varphi\|_{H^{-1/2}(\Gamma)}^2)^{1/2} \). The norm \( \| (u', \varphi) \|_{\mathcal{H}} \) is generated by the bilinear form

\[
a(u, \varphi; v, \psi) := \langle Wu, v \rangle + \langle V \varphi, \psi \rangle.
\]

3.2.1 A posteriori error estimate for the \( h \) version

Let the finite dimensional spaces \( \sigma_h \) and \( \tau_h \) be defined as in (3.15) and (3.16), respectively.

For all \( e \in \omega_h \) let \( \nu_e \) be the middle point of element \( e \). For each of these new nodes \( \nu_e \), \( e \in \omega_h \), we define the piecewise linear basis functions \( b_{h,e} \) by

\[
b_{h,e}(x) = \begin{cases} 
0, & x \in \partial e, \\
1, & x = \nu_e.
\end{cases}
\]

With \( \sigma_e = \text{span}\{b_{h,e}\} \) we denote the one-dimensional space which is spanned by \( b_{h,e} \).

For \( e \in \gamma_h \) we bisect the element \( e \), i.e., \( e = \mu_{e,1} \cup \mu_{e,2} \) and \( |\mu_{e,1}| = |\mu_{e,2}| \). With each \( e \) we associate the Haar basis function \( \beta_{h,e} \) defined as

\[
\beta_{h,e}(x) := \begin{cases} 
1, & \text{if } x \in \bar{\mu}_{e,1}, \\
-1, & \text{if } x \in \bar{\mu}_{e,2}, \\
0, & \text{if } x \in \Gamma \setminus e
\end{cases}
\]

and the one-dimensional space \( \tau_e = \text{span}\{\beta_{h,e}\} \).
Hence we obtain the following subspace decompositions

\[ \sigma_{h/2} = \sigma_h + L_h, \quad L_h := \sum_{e \in \omega_h} \sigma_e, \quad (3.46) \]
\[ \tau_{h/2} = \tau_h + \lambda_h, \quad \lambda_h := \sum_{e \in \gamma_h} \tau_e. \quad (3.47) \]

Let \( P_h : \sigma_{h/2} \to \sigma_h, P_{h,e} : \sigma_{h/2} \to \sigma_e, p_h : \tau_{h/2} \to \tau_h \) and \( p_{h,e} : \tau_{h/2} \to \tau_e \) be the Galerkin projections with respect to the bilinear forms \( \{W, \cdot\} \) and \( \{\cdot, \cdot\} \), respectively. For all \( u \in \sigma_{h/2} \) we define \( P_h \) and \( P_{h,e} \) by

\[ \langle WP_h u, v \rangle = \langle W u, v \rangle \quad \forall v \in \sigma_h, \quad (3.48) \]
\[ \langle WP_{h,e} u, v \rangle = \langle W u, v \rangle \quad \forall v \in \sigma_e, e \in \omega_h, \quad (3.49) \]

and for all \( \varphi \in \tau_{h/2} \) we define \( p_h \) and \( p_{h,e} \) by

\[ \langle Vp_h \varphi, \phi \rangle = \langle V \varphi, \phi \rangle \quad \forall \phi \in \tau_h, \quad (3.50) \]
\[ \langle Vp_{h,e} \varphi, \phi \rangle = \langle V \varphi, \phi \rangle \quad \forall \phi \in \tau_e, e \in \gamma_h. \quad (3.51) \]

Finally, we introduce the two-level Schwarz operators

\[ P := P_h + \sum_{e \in \omega_h} P_{h,e} \quad \text{and} \quad p := p_h + \sum_{e \in \gamma_h} p_{h,e}. \quad (3.52) \]

The following lemma states that the operators \( P \) and \( p \) have bounded condition numbers.

**Lemma 3.3** \([78, 93]\) There are constants \( C_1, C_2 > 0 \) and \( c_1, c_2 > 0 \) independent of \( h \) such that

\[ C_1 \|v\|^2_W \leq \|P_h v\|^2_W + \sum_{e \in \omega_h} \|P_{h,e} v\|^2_W \leq C_2 \|v\|^2_W \quad \forall v \in \sigma_{h/2}, \]
\[ c_1 \|\psi\|^2_V \leq \|p_h \psi\|^2_V + \sum_{e \in \gamma_h} \|p_{h,e} \psi\|^2_V \leq c_2 \|\psi\|^2_V \quad \forall \psi \in \tau_{h/2}. \]

As for finite elements (see, e.g., \([7],[8]\)) we make the saturation assumption:

**Assumption 3.1** Let \( (u, \varphi), (u_h, \varphi_h) \) be the solutions of (3.4) and (3.19), respectively, and let \( (u_{h/2}, \varphi_{h/2}) \) be the solution of (3.19), where every mesh element has been split into two equal parts. Let there exist a constant \( 0 \leq \kappa < 1 \) and a parameter \( h_0 > 0 \) such that for all meshes with mesh parameter \( h \leq h_0 \) (largest element size) there holds

\[ \| (u - u_{h/2}, \varphi - \varphi_{h/2}) \|_H \leq \kappa \| (u - u_h, \varphi - \varphi_h) \|_H. \quad (3.53) \]

**Remark 3.3** We have \( u \in K^\Gamma \subseteq H^{1/2}(\Gamma) \) and \( u_{h/2} \in K_{h/2}^\Gamma \) and \( u_h \in K_h^\Gamma \). There hold the continuous Dirichlet condition \( u_D = g \) and the discrete Dirichlet condition \( u_h(x) = g(x) \forall x \in G_h \). We also have that the "endpoints" \( \Gamma_N \cap \Gamma_D \) and \( \Gamma_S \cap \Gamma_D \) are nodes of the partition \( \omega_h \). Due to the use of Gauss-Lobatto nodes in the construction of \( G_h \) we have that the endpoints are in \( G_h \), therefore there are several points where \( u \) and \( u_h \) are identical, i.e., the difference vanishes. The same argument applies to the difference \( u_{h/2} - u_h \) due to the hierarchical construction of \( \omega_{h/2} \). Therefore the free constant in \( \| \cdot \|_W \) is fixed and \( \| \cdot \|_W \) is indeed a norm.
From the saturation assumption and the triangle inequality we obtain immediately the following result.

**Lemma 3.4** Under Assumption 3.1 the following inequalities hold for all $h \leq h_0$.

\[
(1 - \kappa)\|(u - u_h, \varphi - \varphi_h)\|_H \leq \|(u_{h/2} - u, \varphi_{h/2} - \varphi_h)\|_H \\
\leq (1 + \kappa)\|(u - u_h, \varphi - \varphi_h)\|_H.
\]

Next, we call $\sigma_{h/2}$ the refinement of all elements in $\sigma_h$ and $K^\Gamma_{h/2}$ the corresponding convex subset of $\sigma_h$.

**Theorem 3.6** Let Assumption 3.1 hold. Let $(u_h, \varphi_h)$ be the solution of (3.12) on $K^\Gamma_h \times \tau_h$. Then there are constants $\zeta_1, \zeta_2 > 0$ such that

\[
\zeta_1 \eta_h \leq \|(u - u_h, \varphi - \varphi_h)\|_H \leq \zeta_2 \eta_h
\]

where

\[
\eta_h := (\Theta^2_h + \eta_{u,h}^2 + \eta_{\varphi,h}^2)^{1/2}, \quad \Theta_h := \|P_h e_{h/2}\|_W, \\
\eta_{u,h} := \left(\sum_{e \in \Omega_h} \Theta^2_{h,e}\right)^{1/2}, \quad \Theta_{h,e} := \|P_{h,e} e_{h/2}\|_W, \\
\eta_{\varphi,h} := \left(\sum_{e \in \gamma_h} \theta^2_{h,e}\right)^{1/2}, \quad \theta_{h,e} := \frac{|B(u_h, \varphi_h; 0, \beta_{h,e})|}{\|\beta_{h,e}\|_V}(3.56)
\]

and $e_{h/2} \in K_{u_h} := \{v - u_h \mid v \in K^\Gamma_{h/2}\} = K^\Gamma_{h/2} - u_h$ is the solution of the variational inequality

\[
\langle W e_{h/2}, v - e_{h/2} \rangle \geq \mathcal{L}(v - e_{h/2}, 0) - B(u_h, \varphi_h; v - e_{h/2}, 0) \quad \forall v \in K_{u_h}. (3.58)
\]

**Proof.** For the defect $(\bar{e}_{h/2}, \bar{\varphi}_{h/2}) := (u_{h/2} - u_h, \varphi_{h/2} - \varphi_h)$ of the solution to

\[
B(u_{h/2}, \varphi_{h/2}; v - u_{h/2}, \psi) \geq \mathcal{L}(v - u_{h/2}, \psi) \quad \forall (v, \psi) \in K^\Gamma_{h/2} \times \tau_{h/2}(3.59)
\]

we can write

\[
B(\bar{e}_{h/2}, \bar{\varphi}_{h/2}; v - \bar{e}_{h/2}, \psi) \geq \mathcal{L}(v - \bar{e}_{h/2}, \psi) - B(u_h, \varphi_h; v - \bar{e}_{h/2}, \psi) \quad \forall (v, \psi) \in K_{u_h} \times \tau_{h/2}. (3.60)
\]

We define $(e_{h/2}, \varphi_{h/2}) \in K_{u_h} \times \tau_{h/2}$ by

\[
a(e_{h/2}, \varphi_{h/2}; v - e_{h/2}, \psi) \geq \mathcal{L}(v - e_{h/2}, \psi) - B(u_h, \varphi_h; v - e_{h/2}, \psi) \quad \forall (v, \psi) \in K_{u_h} \times \tau_{h/2}. (3.61)
\]

This variational inequality can be separated into an inequality and an equality which are independent, i.e., (3.61) is equivalent to

\[
\langle We_{h/2}, v - e_{h/2} \rangle \geq \mathcal{L}(v - e_{h/2}, 0) - B(u_h, \varphi_h; v - e_{h/2}, 0) \quad \forall v \in K_{u_h}, (3.62)
\]

\[
\langle V\bar{\varphi}_{h/2}, \psi \rangle = \mathcal{L}(0, \psi) - B(u_h, \varphi_h; 0, \psi) \quad \forall \psi \in \tau_{h/2}. (3.63)
\]

Since

\[
\|e_{h/2}, \varphi_{h/2}\|_H^2 = a(e_{h/2}, \varphi_{h/2}; e_{h/2}, \varphi_{h/2}) = \|e_{h/2}\|_W^2 + \|\varphi_{h/2}\|_V^2
\]
we can apply Lemma 3.3 to obtain
\[
C_1 \| \varepsilon_{h/2} \|^2_W + c_1 \| \varepsilon_{h/2} \|^2_V \\
\leq \| (P_h \varepsilon_{h/2}, p_h) \|^2_H + \sum_{e \in \Omega_h} \| P_{h,e} \varepsilon_{h/2} \|^2_W + \sum_{e \in \Gamma_h} \| p_{h,e} \varepsilon_{h/2} \|^2_V \\
\leq C_2 \| \varepsilon_{h/2} \|^2_W + c_2 \| \varepsilon_{h/2} \|^2_V.
\]
(3.64)

By definition of \( p_h \) and the Galerkin orthogonality we have
\[
\| p_h \varepsilon_{h/2} \|^2_V = (V \varepsilon_{h/2}, p_h \varepsilon_{h/2}) = 0
\]
yielding
\[
\| (P_h \varepsilon_{h/2}, p_h \varepsilon_{h/2}) \|^2_H = \| P_h \varepsilon_{h/2} \|^2_W.
\]
(3.66)

Furthermore we have
\[
p_{h,e} \varepsilon_{h/2} = \frac{(V \varepsilon_{h/2}, \beta_{h,e})}{\| \beta_{h,e} \|^2_V} \beta_{h,e} = \frac{(V \varepsilon_{h/2}, \beta_{h,e})}{\| \beta_{h,e} \|^2_V} \beta_{h,e} \\
= \mathcal{L}(0, \beta_{h,e}) - B(u_h, \varphi_h; 0, \beta_{h,e}) \beta_{h,e} = -B(u_h, \varphi_h; 0, \beta_{h,e}) \beta_{h,e}
\]
and hence
\[
\| p_{h,e} \varepsilon_{h/2} \|^2_V = \frac{|B(u_h, \varphi_h; 0, \beta_{h,e})|}{\| \beta_{h,e} \|^2_V} =: \theta_{h,e}.
\]
(3.67)

Combining the estimates we get
\[
\min\{C_1, c_1\} \| (e_{h/2}, \varepsilon_{h/2}) \|^2_H \leq \| P_h \varepsilon_{h/2} \|^2_W + \sum_{e \in \Omega_h} \| P_{h,e} \varepsilon_{h/2} \|^2_W + \sum_{e \in \Gamma_h} \theta_{h,e}^2 \\
\leq \max\{C_2, c_2\} \| (e_{h/2}, \varepsilon_{h/2}) \|^2_H.
\]

Due to Lemma 2.6 we have
\[
\| (e_{h/2}, \varepsilon_{h/2}) \|_H \sim \| (\tilde{e}_{h/2}, \tilde{\varepsilon}_{h/2}) \|_H = \| (u_{h/2} - u_h, \varphi_{h/2} - \varphi_h) \|_H
\]
(3.68)

and therefore
\[
\frac{1}{\sqrt{\max\{C_2, c_2\}}} \left( \| P_h \varepsilon_{h/2} \|^2_W + \sum_{e \in \Omega_h} \| P_{h,e} \varepsilon_{h/2} \|^2_W + \sum_{e \in \Gamma_h} \theta_{h,e}^2 \right)^{1/2} \\
\leq \| (u_{h/2} - u_h, \varphi_{h/2} - \varphi_h) \|_H \\
\leq \frac{1}{\sqrt{\min\{C_1, c_1\}}} \left( \| P_h \varepsilon_{h/2} \|^2_W + \sum_{e \in \Omega_h} \| P_{h,e} \varepsilon_{h/2} \|^2_W + \sum_{e \in \Gamma_h} \theta_{h,e}^2 \right)^{1/2}.
\]

Applying the saturation assumption we finally obtain the assertion of this theorem with
\[
\zeta_1 = \frac{1}{(1 + \kappa) \sqrt{\max\{C_2, c_2\}}} \text{ and } \zeta_2 = \frac{1}{(1 - \kappa) \sqrt{\min\{C_1, c_1\}}}.
\]
(3.69)

□
**Adaptive Algorithm for the h version:** In this section we formulate an adaptive algorithm which uses the error indicators from Theorem 3.6 to generate a sequence of locally refined meshes. The numerical results in Example 3.1 will show that the corresponding solutions satisfy the saturation assumption and the error estimate (3.54). We estimate the global error by

$$\eta_h := \left( \Theta_h^2 + \sum_{e \in \Omega_h} \Theta_e^2 + \sum_{e \in \gamma_h} \theta_e^2 \right)^{1/2}. \quad (3.70)$$

**Algorithm 3.1** Let the parameters $\epsilon > 0$, $0 < \delta < 1$ and initial subdivisions $\omega_{h0}$, $\gamma_{h0}$ of $\Gamma$ be given. With $\sigma_{(h)}_0$ and $\tau_{(h)}_0$ we denote the initial test and trial spaces. For $k = 0, 1, 2, \ldots$

1. Compute the solution $(u_{(h)}_k, \varphi_{(h)}_k) \in K^{r}_h \times \tau_{(h)}_k$ of (3.19).
2. Solve the defect problem (3.58) for $e_h/2$.
3. Compute the error indicator $\Theta_{(h)}_k$.
4. For each boundary element $e \in \omega_{h_k}$ compute the local error indicator $\Theta_{(h)}_{k,e}$.
5. For each boundary element $e \in \gamma_{h_k}$ compute the local error indicator $\theta_{(h)}_{k,e}$.
6. Compute the global error estimate $\eta_{(h)}_k$. Stop if $\eta_{(h)}_k < \epsilon$.
7. Determine $\Theta_{(h)}_{k,e'}$ such that card($\{e \in \omega_{h_k} : \Theta_{(h)}_{k,e} < \Theta_{(h)}_{k,e'}\}$) = $\delta \text{card}(\omega_{h_k})$ and determine $\theta_{(h)}_{k,e''}$ such that card($\{e \in \gamma_{h_k} : \theta_{(h)}_{k,e} < \theta_{(h)}_{k,e''}\}$) = $\delta \text{card}(\gamma_{h_k})$.

Refine the boundary element $e \in \omega_{h_k}$ iff $\Theta_{(h)}_{k,e} \geq \Theta_{(h)}_{k,e'}$ and refine the boundary element $e \in \gamma_{h_k}$ iff $\theta_{(h)}_{k,e} \geq \theta_{(h)}_{k,e''}$. This defines new meshes $\omega_{h_{k+1}}$, $\gamma_{h_{k+1}}$ and enlarged spaces $\sigma_{(h)}_{k+1} \supset \sigma_{(h)}_k$, $\tau_{(h)}_{k+1} \supset \tau_{(h)}_k$. Goto 1.

The above refinement strategy ensures that at least $100 \cdot (1 - \delta)$ percent of the elements will be refined.

### 3.2.2 A posteriori error estimate for the hp version

Let the finite dimensional spaces $\sigma_{hp}$ and $\tau_{hp}$ be defined as in (3.8) and (3.9), respectively. We extend the spaces $\sigma_{hp}$ and $\tau_{hp}$ by bubble functions given on each element in $\omega_h$ and $\gamma_h$.

Each element $e \in \omega_h$ is associated with a polynomial degree $p_e$ and the affine mapping $F_e$ with $F_e(\xi) = x(\xi) \in e$ for $\xi \in [-1, 1]$.

With the Legendre polynomial $L_j$ of degree $j$ we set

$$\psi_{0}(\xi) := \frac{1 - \xi}{2}, \quad \psi_{1}(\xi) := \frac{1 - \xi}{2}, \quad \psi_{j}(\xi) := \sqrt{\frac{2j - 1}{2} \int_{-1}^{\xi} L_{j-1}(t) \, dt}, \quad 2 \leq j,$$

and take

$$\sigma_{e} = \text{span}\{\psi_{e,p_e+1}\}, \quad \psi_{e,p_e+1}(x) := \psi_{p_e+1}(F_e^{-1}(x)). \quad (3.71)$$

On the other hand, each element $e \in \gamma_h$ is associated with a polynomial degree $q_e$. Setting

$$\phi_{0}(t) = \frac{1}{2}, \quad \phi_{j}(t) := \sqrt{\frac{2j + 1}{2} L_{j}(t)}, \quad 1 \leq j,$$
we take
\[ \tau_e = \text{span}\{\phi_{e,q_e+1}\}, \quad \phi_{e,q_e+1}(x) := \phi_{e,q_e+1}(F_e^{-1}(x)). \] (3.72)

Hence we obtain the subspace decompositions
\[ \sigma_{h,p+1} := \sigma_{hp} \oplus L_p, \quad L_p := \sum_{e \in \omega_h} \sigma_e, \] (3.73)
\[ \tau_{h,p+1} := \tau_{hp} \oplus \lambda_p, \quad \lambda_p := \sum_{e \in \gamma_h} \tau_e. \] (3.74)

Hence the polynomial degree vector associated with \( \sigma_{h,p+1} \) is \( (p_e + 1)_{e \in \omega_h} \) and the polynomial degree vector associated with \( \tau_{h,p+1} \) is \( (q_e + 1)_{e \in \gamma_h} \).

Let \( P_{hp} : \sigma_{h,p+1} \rightarrow \sigma_{hp} \), \( P_{hp,e} : \sigma_{h,p+1} \rightarrow \sigma_e \), \( p_{hp} : \tau_{h,p+1} \rightarrow \tau_{hp} \), \( p_{hp,e} : \tau_{h,p+1} \rightarrow \tau_e \) be the Galerkin projections with respect to the bilinear forms \( \langle W \cdot , \cdot \rangle \) and \( \langle V \cdot , \cdot \rangle \). For all \( u \in \sigma_{h,p+1} \) we define \( P_{hp} \) and \( P_{hp,e} \) by
\[ \langle WP_{hp} u, v \rangle = \langle Wu, v \rangle \quad \forall v \in \sigma_{hp}, \] (3.75)
\[ \langle WP_{hp,e} u, v \rangle = \langle Wu, v \rangle \quad \forall v \in \sigma_e, e \in \omega_h, \] (3.76)

and for all \( \varphi \in \tau_{h,p+1} \) we define \( p_{hp} \) and \( p_{hp,e} \) by
\[ \langle Vp_{hp} \varphi, \phi \rangle = \langle V \varphi, \phi \rangle \quad \forall \phi \in \tau_{hp}, \] (3.77)
\[ \langle Vp_{hp,e} \varphi, \phi \rangle = \langle V \varphi, \phi \rangle \quad \forall \phi \in \tau_e, e \in \gamma_h. \] (3.78)

Finally, we define the two-level Schwarz operators
\[ P_\sigma := P_{hp} + \sum_{e \in \omega_h} P_{hp,e} \quad \text{and} \quad p_r := p_{hp} + \sum_{e \in \gamma_h} p_{hp,e}. \] (3.79)

The following lemma states that the operators \( P_\sigma \) and \( p_r \) have condition numbers depending only logarithmically on \( p_{max} \) and are independent of \( h \).

**Lemma 3.5** There exist constants \( c_1, c_2 > 0 \) and \( C_1, C_2 > 0 \) independent of \( h \) and \( p = (p_e)_{e \in \omega_h}, q = (q_e)_{e \in \gamma_h} \) such that there hold
\[ C_1(1 + \log p_{max})^{-2} \| v \|_W^2 \leq \| P_{hp} v \|_W^2 + \sum_{e \in \omega_h} \| P_{hp,e} v \|_W^2 \leq C_2 \| v \|_W^2 \quad \forall v \in \sigma_{h,p+1}, \] (3.80)
\[ c_1(1 + \log q_{max})^{-2} \| \psi \|_V^2 \leq \| p_{hp} \psi \|_V^2 + \sum_{e \in \gamma_h} \| p_{hp,e} \psi \|_V^2 \leq c_2 \| \psi \|_V^2 \quad \forall \psi \in \tau_{h,p+1} \] (3.81)

with \( p_{max} := \max\{p_e, e \in \omega_h\} \) and \( q_{max} := \max\{q_e, e \in \gamma_h\} \).

**Proof.** Due to \cite[Lemma 3.2]{54} there holds
\[ C_1(1 + \log p_{max})^{-2} (\| v_{hp} \|_W^2 + \sum_{e \in \omega_h} \| v_e \|_W^2) \leq \| v \|_W^2 \leq C_2 (\| v_{hp} \|_W^2 + \sum_{e \in \omega_h} \| v_e \|_W^2) \] (3.82)

for all \( v = v_{hp} + \sum_{e \in \omega_h} v_e \in \sigma_{h,p+1} \) where \( v_{hp} \in \sigma_{hp} \) and \( v_e \in \sigma_e, e \in \omega_h. \) Due to the definition of \( P_\sigma \) there holds
\[ \langle WP_\sigma v, v \rangle = \langle WP_{hp} v, v \rangle + \sum_{e \in \omega_h} \langle WP_{hp,e} v, v \rangle \]
\[ = \langle WP_{hp} v, P_{hp} v \rangle + \sum_{e \in \omega_h} \langle WP_{hp,e} v, P_{hp,e} v \rangle = \| P_{hp} v \|_W^2 + \sum_{e \in \omega_h} \| P_{hp,e} v \|_W^2 \]
for all $v \in \sigma_{h,p+1}$.

Analogously, there holds due to [54, Proof of Theorem 4.1]

$$c_1(1 + \log q_{\text{max}})^{-2}(\|\psi_{hp}\|_V^2 + \sum_{\omega \in \gamma_h} ||\psi_{\omega}||_V^2 \leq \|\psi\|_V^2 \leq c_2(\|\psi_{hp}\|_V^2 + \sum_{\omega \in \gamma_h} ||\psi_{\omega}||_V^2))$$

(3.83)

for all $\psi = \psi_{hp} + \sum_{\omega \in \gamma_h} \psi_{\omega} \in \Gamma_{h,p+1}$ where $\psi_{hp} \in \Gamma_{hp}$ and $\psi_{\omega} \in \Gamma_{\omega}$, $\omega \in \gamma_h$. Due to the definition of $P_\tau$ there holds

$$\langle V P_\tau \psi, \psi \rangle = ||P_{hp}\psi||_V^2 + \sum_{\omega \in \gamma_h} ||P_{hp,\omega}\psi||_V^2$$

for all $\psi \in \Gamma_{h,p+1}$. \hfill \square

Let $(u, \varphi)$ be the solution of the variational inequality (3.4) and let $(u_{hp}, \varphi_{hp}) \in \sigma_{hp} \times \tau_{hp}$, $(u_{h,p+1}, \varphi_{h,p+1}) \in \sigma_{h,p+1} \times \tau_{h,p+1}$ be the solutions of the corresponding discrete problems. As for finite element problems (see, e.g., [8], [7]) we make the saturation assumption:

**Assumption 3.2** There exists a parameter $0 \leq \kappa < 1$ such that for all discrete spaces holds:

$$\|(u - u_{h,p+1}, \varphi - \varphi_{h,p+1})\|_{\mathcal{W}} \leq \kappa \|(u - u_{hp}, \varphi - \varphi_{hp})\|_{\mathcal{W}}.$$  

(3.84)

**Remark 3.4** The arguments from Remark 3.3 apply here, too. Therefore we can use $\| \cdot \|_{\mathcal{W}}$ to describe the error.

**Theorem 3.7** We assume that (3.84) holds. Then, there are constants $\zeta_1, \zeta_2 > 0$ such that

$$\zeta_1 \eta_{hp} \leq \|(u - u_{hp}, \varphi - \varphi_{hp})\|_{\mathcal{W}} \leq \zeta_2 (1 + \log \max\{p_{\text{max}}, q_{\text{max}}\}) \eta_{hp}$$

(3.85)

where

$$\eta_{hp} := \left(\Theta_{hp}^2 + \eta_{a,h,p}^2 + \eta_{e,h,p}^2\right)^{1/2}, \quad \Theta_{hp} := ||P_{hp}e_{hp,p+1}\|_{W},$$

(3.86)

$$\eta_{a,h,p} := \left(\sum_{\omega \in \omega_h} \Theta_{hp,\omega}^2\right)^{1/2}, \quad \Theta_{hp,\omega} := ||P_{hp,\omega}e_{hp,p+1}\|_{W},$$

(3.87)

$$\eta_{e,h,p} := \left(\sum_{\omega \in \gamma_h} \Theta_{hp,\omega}^2\right)^{1/2}, \quad \Theta_{hp,\omega} := \frac{|B(u_{hp}, \varphi_{hp}; \cdot, \psi_{e,h,p+1})|}{\|\psi_{e,h,p+1}\|_V},$$

(3.88)

and $e_{hp,p+1} \in K_{uhp}$ is the solution of the variational inequality

$$\langle W_{e_{h,p+1}}, v - e_{h,p+1}\rangle \geq \mathcal{L}(v - e_{h,p+1}, 0) - B(u_{hp}, \varphi_{hp}; v - e_{h,p+1}, 0) \quad \forall v \in K_{uhp}.$$  

(3.89)

**Proof.** For the defect $(\tilde{e}_{h,p+1}, \tilde{\varphi}_{h,p+1}) := (u_{h,p+1} - u_{hp}, \varphi_{h,p+1} - \varphi_{hp})$ of the solution to

$$B(u_{h,p+1}, \varphi_{h,p+1}; v - u_{h,p+1}, \psi) \geq \mathcal{L}(v - u_{h,p+1}, \psi) \quad \forall (v, \psi) \in K_{h,p+1} \times \tau_{h,p+1}$$

(3.90)

we can write

$$B(\tilde{e}_{h,p+1}, \tilde{\varphi}_{h,p+1}; v - \tilde{e}_{h,p+1}, \psi) \geq \mathcal{L}(v - \tilde{e}_{h,p+1}, \psi) - B(u_{hp}, \varphi_{hp}; v - \tilde{e}_{h,p+1}, \psi)$$

(3.91)

for all $(v, \psi) \in K_{uhp} \times \tau_{h,p+1}$.

We define $(\tilde{e}_{h,p+1}, \tilde{\varphi}_{h,p+1}) \in K_{uhp} \times \tau_{h,p+1}$ by

$$a(e_{h,p+1}, \varphi_{h,p+1}; v - e_{h,p+1}, \psi) \geq \mathcal{L}(v - e_{h,p+1}, \psi) - B(u_{hp}, \varphi_{hp}; v - e_{h,p+1}, \psi)$$

(3.92)
for all \((v, \psi) \in K_{u_{hp}} \times \tau_{h,p+1}\).

This variational inequality can be separated into an inequality and an equality which are independent, i.e., (3.92) is equivalent to

\[
\langle We_{h,p+1}, v - e_{h,p+1} \rangle \geq L(v - e_{h,p+1}, 0) - B(u_{hp}, \varphi_{hp}; v - e_{h,p+1}, 0) \quad \forall v \in K_{u_{hp}},
\]

\[
\langle V\varepsilon_{h,p+1}, \psi \rangle = L(0, \psi) - B(u_{hp}, \varphi_{hp}; 0, \psi) \quad \forall \psi \in \tau_{h,p+1}.
\]

(3.93)

(3.94)

Since

\[
\|(e_{h,p+1}, \varepsilon_{h,p+1})\|_H^2 = a(e_{h,p+1}, \varepsilon_{h,p+1}; e_{h,p+1}, \varepsilon_{h,p+1}) = \|e_{h,p+1}\|_W^2 + \|\varepsilon_{h,p+1}\|_V^2,
\]

we can apply Lemma 3.5 to obtain

\[
C_1(1 + \log p_{max})^{-2}\|e_{h,p+1}\|_W^2 + C_1(1 + \log q_{max})^{-2}\|\varepsilon_{h,p+1}\|_V^2 \\
\leq \|(P_{hp}e_{h,p+1}, P_{hp}\varepsilon_{h,p+1})\|_H^2 + \sum_{e \in \omega_h} \|P_{hp,e}e_{h,p+1}\|_W^2 + \sum_{e \in \gamma_h} \|P_{hp,e}\varepsilon_{h,p+1}\|_V^2 \\
\leq C_2\|e_{h,p+1}\|_W^2 + c_2\|\varepsilon_{h,p+1}\|_V^2.
\]

(3.95)

By definition of \(p_{hp}\) and the Galerkin orthogonality we have

\[
\|(P_{hp}e_{h,p+1}, P_{hp}\varepsilon_{h,p+1})\|_H^2 = \langle V\varepsilon_{h,p+1}, p_{hp}e_{h,p+1} \rangle = 0
\]

(3.96)

yielding

\[
\|(P_{hp}e_{h,p+1}, P_{hp}\varepsilon_{h,p+1})\|_H^2 = \|P_{hp}e_{h,p+1}\|_W^2.
\]

(3.97)

Furthermore

\[
P_{hp,e}\varepsilon_{h,p+1} = \frac{\langle V\varepsilon_{h,p+1}, \phi_{e,q_e+1} \rangle}{\langle V\phi_{e,q_e+1}, \phi_{e,q_e+1} \rangle} \phi_{e,q_e+1} = \frac{\langle V\varepsilon_{h,p+1}, \phi_{e,q_e+1} \rangle}{\|\phi_{e,q_e+1}\|_V^2} \phi_{e,q_e+1} \\
= \frac{L(0, \phi_{e,q_e+1}) - B(u_{hp}, \varphi_{hp}; 0, \phi_{e,q_e+1})}{\|\phi_{e,q_e+1}\|_V^2} \phi_{e,q_e+1} \\
= -\frac{B(u_{hp}, \varphi_{hp}; 0, \phi_{e,q_e+1})}{\|\phi_{e,q_e+1}\|_V^2} \phi_{e,q_e+1}
\]

and hence

\[
\|P_{hp,e}\varepsilon_{h,p+1}\|_V = \frac{|B(u_{hp}, \varphi_{hp}; 0, \phi_{e,q_e+1})|}{\|\phi_{e,q_e+1}\|_V} =: \theta_{hp,e}.
\]

(3.98)

Combining the estimates we get

\[
c_3\|(e_{h,p+1}, \varepsilon_{h,p+1})\|_H^2 \leq \|P_{hp}e_{h,p+1}\|_W^2 + \sum_{e \in \omega_h} \|P_{hp,e}e_{h,p+1}\|_W^2 + \sum_{e \in \gamma_h} \theta_{hp,e}^2 \\
\leq c_4\|(e_{h,p+1}, \varepsilon_{h,p+1})\|_H^2
\]

with \(c_3 := \min\{C_1(1 + \log p_{max})^{-2}, c_1(1 + \log q_{max})^{-2}\} \geq \min\{C_1, c_1\}(1 + \log \max\{p_{max}, q_{max}\})^{-2}\)

and \(c_4 := \max\{C_2, c_2\}\).

Due to Lemma 2.6 we have

\[
\|(e_{h,p+1}, \varepsilon_{h,p+1})\|_H \sim \|(\tilde{e}_{h,p+1}, \tilde{\varepsilon}_{h,p+1})\|_H = \|(u_{hp+1} - u_{hp}, \varphi_{h,p+1} - \varphi_{hp})\|_H
\]

(3.99)
Applying the saturation assumption we finally obtain the assertion of this theorem with

\[
\frac{1}{\sqrt{C_4}} \left( \| P_{hp} e_{h,p+1} \|^2_W + \sum_{e \in \omega_h} \| P_{hp} e_{h,p+1} \|^2_W + \sum_{e \in \gamma_h} \theta^2_{hp,e} \right)^{1/2} \\
\leq \| (u_{hp+1} - u_{hp}, \varphi_{hp+1} - \varphi_{hp}) \|_H \\
\leq \frac{1}{\sqrt{C_3}} \left( \| P_{hp} e_{h,p+1} \|^2_W + \sum_{e \in \omega_h} \| P_{hp} e_{h,p+1} \|^2_W + \sum_{e \in \gamma_h} \theta^2_{hp,e} \right)^{1/2}.
\]

Applying the saturation assumption we finally obtain the assertion of this theorem with

\[
\zeta_1 = \frac{1}{(1 + \kappa) \sqrt{\max\{C_2, c_2\}}} \quad \text{and} \quad \zeta_2 = \frac{1}{(1 - \kappa) \sqrt{\min\{C_1, c_1\}}}, \quad (3.100)
\]

\[\square\]

**Adaptive Algorithm for the hp version:** Now, we formulate an hp-adaptive three-step algorithm which uses the error indicators from Theorem 3.7 to generate a sequence of locally refined meshes.

We neglect the factor \((1 + \log \max\{p_{\max}, q_{\max}\})^2\) in Theorem 3.7 which contains no information suitable for local refinement and estimate the global error by

\[
\eta_{hp} := \left( \Theta^2_{hp} + \sum_{e \in \omega_h} \Theta^2_{hp,e} + \sum_{e \in \gamma_h} \theta^2_{hp,e} \right)^{1/2}. \quad (3.101)
\]

**Algorithm 3.2** Let the parameters \(\epsilon > 0\), \(0 < \delta_1 < \delta_2 < 1\) and initial subdivisions \(\omega_{h_0}, \gamma_{h_0}\) of \(\Gamma\) and initial polynomial degree vectors \(p_0, q_0\) be given. With \(\sigma_{(hp)_0}\) and \(\tau_{(hp)_0}\) we denote the initial test and trial spaces. For \(k = 0, 1, 2, \ldots\)

1. Compute the solution \((u_{(hp)_k}, \varphi_{(hp)_k}) \in K_{(hp)_k}^\Gamma \times \tau_{hp}\) of (3.12).
2. Solve the defect problem (3.89) for \(e_{h,p+1}\).
3. Compute the error indicator \(\Theta_{(hp)_k}\).
4. For each boundary element \(e \in \omega_{h_k}\) compute the local error indicator \(\Theta_{hp,e}\).
5. For each boundary element \(e \in \gamma_{h_k}\) compute the local error indicator \(\theta_{hp,e}\).
6. Compute the global error estimate \(\eta_{(hp)_k}\). Stop if \(\eta_{(hp)_k} < \epsilon\).
7. Determine \(\Theta_{(hp)_k,e_i}\) such that \(\text{card}\{e \in \omega_{h_k} : \Theta_{(hp)_k,e} < \Theta_{(hp)_k,e_i}\} = \delta_i \text{card}(\omega_{h_k})\) for \(i = 1, 2\) and determine \(\theta_{(hp)_k,e_i'}\) such that \(\text{card}\{e \in \gamma_{h_k} : \theta_{(hp)_k,e} < \theta_{(hp)_k,e_i'}\} = \delta_i \text{card}(\gamma_{h_k})\) for \(i = 1, 2\).
8. Adaption step.

**h-adaptive algorithm:** If \(\Theta_{(hp)_k,e} \geq \Theta_{(hp)_k,e_i}\), split the element \(e \in \omega_{h_k}\) into two elements of the same size and inherit the polynomial degree. If \(\theta_{(hp)_k,e} \geq \theta_{(hp)_k,e_i}\), split the element \(e \in \gamma_{h_k}\) into two elements of the same size and inherit the polynomial degree.
p–adaptive algorithm: If \( \Theta_{(hp)}(e) \geq \Theta_{(hp)}(e'), \) increase the polynomial degree on \( e \in \omega_{h_k} \) by 1. If \( \theta_{(hp)}(e) \geq \theta_{(hp)}(e')', \) increase the polynomial degree on \( e \in \gamma_{h_k} \) by 1.

hp–adaptive algorithm: If \( \Theta_{(hp)}(e') \geq \Theta_{(hp)}(e) \geq \Theta_{(hp)}(e')', \) increase the polynomial degree on \( e \in \omega_{h_k} \) by 1. If \( \Theta_{(hp)}(e') > \Theta_{(hp)}(e) \), split the element \( e \in \omega_{h_k} \) into two elements of the same size and inherit the polynomial degree. If \( \Theta_{(hp)}(e) < \Theta_{(hp)}(e')', \) do nothing.

This defines new meshes \( \omega_{h_{k+1}}, \gamma_{h_{k+1}} \) and enlarged spaces \( \sigma_{(hp)}(k+1) \supset \sigma_{(hp)}(k), \tau_{(hp)}(k+1) \supset \tau_{(hp)}(k) \).

The above refinement strategy ensures that at least \( 100 \cdot (1 - \delta_1) \) percent of the elements will be refined.

Remark 3.5 The h–adaptive algorithm described above corresponds to Algorithm 3.1. Instead of the enriched trial space containing additional hat-functions and derivatives of hat-functions (i.e., jump-functions) described there, we are now using bubble functions.

Remark 3.6 The algorithms 3.1 and 3.2 can be simplified, if we demand that there always holds \( \omega_h = \gamma_h \). We still compute both sets of error indicators, but both refinement criteria will be applied simultaneously to \( \gamma_h \).

Remark 3.7 In every adaptive step the auxiliary defect problem (3.89) has to be solved. Due to the temporarily increased polynomial degree this defect problem has at most twice as many unknowns as the original problem. For higher polynomial degrees this effect becomes smaller. Fortunately, there is no need to compute the action of the Schur complement as in the implementation of the discrete Steklov-Poincaré operator, therefore the time for solving this auxiliary problem is comparable to the time for solving the original problem.
3.3 Solvers for BEM problems

3.3.1 SOR with projection for BEM problems

In case of the two-dimensional Lamé problem presented in Section 3.1.3 the projection to $K_N$ is defined by

$$\text{Pr}_{K_N} := \begin{cases} \mathbb{R}^{2N} & \rightarrow K_N \\ \bar{u} & \mapsto \text{Pr}_{K_N} \bar{u} \text{ with } \|\bar{u} - \text{Pr}_{K_N} \bar{u}\|_2 = \inf_{\bar{v} \in K_N} \|\bar{u} - \bar{v}\|_2 \end{cases}$$

and can be computed in case of the $h$ version by

$$(\text{Pr}_{K_N}(\bar{u}))_i := \begin{cases} \bar{u}_i & \text{if } p_i \in G_h \cap \Gamma_N \\ g(p_i) & \text{if } p_i \in G_h \cap \Gamma_D \\ \tilde{t}_i u_{n,i} + n_i \min(u_{n,i}, g_{n,i}) & \text{if } p_i \in G_h \cap \Gamma_S \end{cases}$$

where $p_i$ are the nodes of $\omega_h$ and $g_{n,i} = g_n(p_i)$. The projection for the Laplace problem is defined analogously.

We know that the matrix representation of the Steklov-Poincaré operator $S_N$ is positive semi-definite and symmetric. Therefore the Gauß-Seidel over-relaxation algorithm (SOR) with projection (see Algorithm 2.1) is globally convergent. But the only advantage of this algorithm is that it is easily implemented apart from global convergence. The drawbacks are its very slow convergence and its computational costs. First, we have to compute a Schur complement, involving the inversion of a dense matrix and two dense matrix-matrix multiplications. Second, in every iteration we have to invest the computational costs of a dense matrix-vector multiplication.

Due to simplicity in the following we take only the Laplace operator into account. The Lamé operator can be handled analogously.

3.3.2 Preconditioners for the Polyak algorithm

An algorithm with a (possible) better performance than SOR is the preconditioned Polyak algorithm (see Algorithm 2.2). The unmodified algorithm still needs the computation of a Schur complement, but at least its asymptotical convergence rate only depends on the condition number of the preconditioned matrix.

We know that the norm defined by the discrete Steklov-Poincaré operator $S_N$ is equivalent to the $H^{1/2}(\Gamma)/\mathbb{R}$ norm. Therefore all preconditioners for the hypersingular integral operator, with a proof of convergence based on the equivalence of the norm defined by the hypersingular integral operator with the $H^{1/2}(\Gamma)/\mathbb{R}$ norm, are immediately applicable to our discrete Steklov-Poincaré operator. In the following we collect some of these preconditioners.

**Theorem 3.8** Let $P_{\text{Wave}}$ be the additive Schwarz operator corresponding to the pre-wavelets given in [24].

There exist positive constants $C_1$ and $C_2$ such that

$$C_1 \langle S_h v, v \rangle \leq \langle S_h P_{\text{Wave}} v, v \rangle \leq C_2 \langle S_h v, v \rangle \text{ for any } v \in S^L = \sigma_h \text{ with } \dim S^L = 2^L \quad (3.102)$$

otherwise

$$C_1 \langle S_h v, v \rangle \leq \langle S_h P_{\text{Wave}} v, v \rangle \leq C_2 \langle 1 + \log \frac{1}{h} \rangle \langle S_h v, v \rangle \text{ for any } v \in S^L = \sigma_h. \quad (3.103)$$
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**Proof.** See the proof for the hypersingular integral operator in Tran, Stephan, Zaprianov [95].

**Remark 3.8** In the case of the weakly singular operator there holds an analogous theorem if we define our prewavelets now as derivatives of the continuous prewavelets defined before, see also Tran, Stephan, Zaprianov [95].

**Theorem 3.9** Let $P_{\text{Hier}}$ be the additive Schwarz operator given by the hierarchical base splitting in Yserentant [102].

There exist positive constants $C_1$ and $C_2$ such that

$$C_1(1 + \log \frac{1}{h})^2 \langle S_h v, v \rangle \leq \langle S_h P_{\text{Hier}}v, v \rangle \leq C_2 \langle S_h v, v \rangle \quad \text{for any } v \in S^L. \quad (3.104)$$

**Proof.** See [102].

**Theorem 3.10** Let $P_{\text{MG}}$ be one V-cycle multigrid step acting as a preconditioner.

There exist positive constants $C_1$ and $C_2$ such that

$$C_1 \langle S_h v, v \rangle \leq \langle S_h P_{\text{MG}}v, v \rangle \leq C_2 \langle S_h v, v \rangle \quad \text{for any } v \in \sigma_h. \quad (3.105)$$

**Proof.** See Petersdorff, Stephan [97].

**Theorem 3.11** Let $P_{\text{BPX}}$ be the BPX or Multilevel Additive Schwarz preconditioner.

There exist positive constants $C_1$ and $C_2$ such that

$$C_1(1 + (\log \frac{1}{h})^2) \langle S_h v, v \rangle \leq \langle S_h P_{\text{BPX}}v, v \rangle \leq C_2 \langle S_h v, v \rangle \quad \text{for any } v \in \sigma_h. \quad (3.106)$$

**Proof.** See Tran, Stephan [93].

**Remark 3.9** The preconditioners mentioned above usually only depend on the matrix in form of a matrix-vector multiplication (at least they can be modified in this way). The preconditioned Polyak-algorithm also only depends on the matrix in form of a matrix-vector multiplication. Therefore the computation of $y = S_N x$ can be done without the computation of the Schur complement using the following algorithm.

**Algorithm 3.3**

The computation of the Schur complement without explicitly known inverse of $V_N$ reads:

Input $x, W_N, V_N, (I + K)_N$

Solve $V_N z = (I + K)_N x$ (e.g., by the multigrid algorithm)

Compute $y = W_N x + (I + K)_N z$.

Therefore the computation of $y = S_N x$ demands only the computation of 3 matrix-vector multiplications and the solution of one linear system, which can be done very efficiently in $O(N^2)$-time. Therefore, $y = S_N x$ is only a $O(N^2)$-operation. Depending on how much iterations of the preconditioned Polyak-algorithm have to be done, this modification may be much more efficient.
3.3.3 Monotone multigrid methods

Let $\Gamma = \partial \Omega$ be the boundary of a polygonal domain $\Omega$. The Signorini problem is:

Find $u \in K : a(u, v - u) \geq l(v - u), \forall v \in K$

where $K$ is a closed, convex subset $K \subset H^{1/2}(\Gamma)$ of the form

$$K := \{ v \in H^{1/2}(\Gamma) | v(x) \leq g(x) \forall x \in \Gamma_S \text{ and } v(x) = g(x) \forall x \in \Gamma_D \}.$$ 

In the following we will restrict ourself to the part of the variational inequality directly involving the Signorini condition. The Neumann condition can be handled like the Signorini condition by defining $g$ to be very large on $\Gamma_D$ and the part of $a(\cdot, \cdot)$ defined on $\Gamma_D$ can readily absorbed into the linear functional $l(\cdot)$. To simplify the analysis we assume below that $\Gamma$ is an open curve on which the Signorini condition is given.

Let $N_k := \{ p_i^{(k)} : 1 \leq i \leq n_k \}, 0 \leq k \leq j$, denote the set of all mesh points located at $\Gamma$. Here, we assume that these sets are constructed by iteratively splitting all boundary elements into two equal parts, i.e.,

$$p_{2i-1}^{(k+1)} := p_i^{(k)}, \quad p_{2i}^{(k+1)} := (p_i^{(k)} + p_{i+1}^{(k)})/2, \quad 1 \leq i \leq n_k \quad \text{with } p_{n_k+1}^{(k)} := p_1^{(k)}.$$

The boundary $\Gamma$ can be decomposed at all levels in

$$\Gamma = \bigcup_{i=1}^{n_k} \Gamma_i^{(k)}, \quad \text{with } \Gamma_i^{(k)} := p_i^{(k)} p_{i+1}^{(k)}.$$

Then we can define the hat functions $\phi_i^{(k)}$ by

$$\phi_i^{(k)}(p_i^{(k)}) := \begin{cases} 1, & l = i, \\ 0, & l \neq i \end{cases}, \quad i, l = 1, \ldots, n_k, \quad k = 0, \ldots, j, \quad n_k = 2^k \cdot n_0$$

and define

$$S_k := \text{span}\{\phi_i^{(k)} | i = 1, \ldots, n_k\}, \quad k = 0, \ldots, j.$$ 

**Monotone Multigrid for Discrete Formulation II:** We denote by $g_j$ the discrete obstacle at level $j$

$$g_j = \sum_{i=1}^{n_j} g(p_i^{(j)}) \phi_i^{(j)}.$$

For some fixed $k$, $0 \leq k \leq j - 1$, we have a certain order of the nodes $E_k = \{ p_1^k, \ldots, p_{n_k}^k \}$ with midpoints $p_e \in N_{k+1}$.

**Restriction for defect:** Crucial for the convergence of the monotone multigrid method is the restriction of the defect to the coarse grid, because we have to ensure, that our approximation to the solution is always in $K$. We construct the restriction operator $R_{k+1}$ as a multiplicative composition from elementary restriction operators $R_e$.

$$R_{k+1} = I_{S_k} \circ R_{n_{n_k}} \circ p_{n_k}^k \circ \cdots \circ R_{p_1^k} \circ v, \quad v \in S_{k+1}$$

$$R_e : \begin{cases} S_{k+1} & \rightarrow S_{k+1} \\ v & \mapsto R_e v := v + v_1 \phi_{p_1}^{(k+1)} + v_2 \phi_{p_2}^{(k+1)} \end{cases}$$

(3.107) (3.108)
v_1, v_2 \in \mathbb{R} are determined by \( R_v(p) \leq v(p), p = p_1, p_e, p_2 \). A quasi-optimal restriction operator \( R_e \) (cf. Figure 3.4) is defined by

\[
\begin{align*}
v_1 &= \begin{cases} 
0 & \text{if } v(p_1) \leq v(p_e) \text{ or } v(p_1) + v(p_2) \leq 2v(p_e) \\
2v(p_e) - v(p_1) - v(p_2) & \text{if } v(p_2) \leq v(p_e) \leq v(p_1) \\
v(p_e) - v(p_1) & \text{if } v(p_e) \leq v(p), p = p_1, p_2
\end{cases}
\end{align*}
\]

\[
\begin{align*}
v_2 &= \begin{cases} 
0 & \text{if } v(p_2) \leq v(p_e) \text{ or } v(p_1) + v(p_2) \leq 2v(p_e) \\
2v(p_e) - v(p_1) - v(p_2) & \text{if } v(p_1) \leq v(p_e) \leq v(p_2) \\
v(p_e) - v(p_2) & \text{if } v(p_e) \leq v(p), p = p_1, p_2
\end{cases}
\end{align*}
\]

Next we describe the monotone multigrid algorithm, analogously to Kornhuber [63], c.f. [75].

**Algorithm 3.4** The standard monotone multigrid algorithm for the boundary element method on curves reads:

Initial iterate \( w_j \)

Initial defect \( \psi^{(j)} := (g_j - w_j) \)

For level \( k = j, \ldots, 0 \) do

if \( k = j \)

\( l^{(j)}(v) := l(v) - a(w_j, v), \forall v \in S_j \)

else

\( l^{(k)}(v) := l^{(k+1)}(v) - a(w^{(k+1)}, v), \forall v \in S_k \) (Restriction)

endif

Set \( w_0^{(k)} = 0 \)

For \( i = 1, \ldots, n_k \) do

Find \( \text{span}\{\phi_i^{(k)}\} \ni v_i^{(k)} \leq \psi^{(k)}(p_i^k)\phi_i^{(k)} \) with

\[
a(v_i^{(k)}, v - v_i^{(k)}) \geq l_k(v - v_i^{(k)}) - a(v_{i-1}^{(k)}, v - v_i^{(k)}) \quad \forall v \leq \psi^{(k)}(p_i^k)\phi_i^{(k)}
\]

\[
w_i^{(k)} := w_{i-1}^{(k)} + v_i^{(k)}
\]
CHAPTER 3. BEM WITH SIGNORINI CONTACT

\[ w^{(k)} := w^{(k)}_{N_k} \]

\[ \text{if } k \geq 1 \]

\[ \psi^{(k-1)} = R_k(\psi^{(k)} - w^{(k)}) \quad (\text{local defect at level } k - 1) \]

\[ \text{endif} \]

\[ \hat{w}_j = w_j + \sum_{k=0}^j w^{(k)} \quad (\text{Prolongation}) \]

**Theorem 3.12** The monotone multigrid method applied to the boundary problem (S) for the Steklov-Poincaré variational inequality converges globally. It has the asymptotical convergence rate of \( (1 - c(j+1)^{-4}) \) with respect to the energy norm.

**Proof.** We are using the quasi-optimal restriction operator \( R_e \) as originally defined in [63, Section 3.1.3]. The Steklov-Poincaré operator is a pseudodifferential operator of order one, i.e., positive order. Therefore the abstract theory in [63] can be applied and we know due to [63, Theorem 2.9, Proposition 3.3] that the monotone multigrid method is globally convergent and reduces asymptotically to the usual multigrid method on all nodes with \( u(p_i^{(j)}) \neq g(p_i^{(j)}) \ i = 1, \ldots, n_j = 2^j \cdot n_0 \), i.e., all nodes where the solution stays away from the Dirichlet-Signorini conditions. Therefore, we can apply the results on multigrid algorithms without regularity assumptions in [14].

\( \square \)
3.4 Numerical examples for BEM problems

3.4.1 Examples for the Laplace equation

Example 3.1 We consider the Signorini problem (3.1) on the square \( \Omega = [-1/4, 1/4]^2 \), where the horizontal sides give \( \Gamma_N \), the right vertical side \( \Gamma_D \) and the left vertical side \( \Gamma_S \), see Figure 3.5. We choose \( g = 0 \) on \( \Gamma_D \) and \(-3/4\) on \( \Gamma_S \) and \( h = n_x + n_y \) on \( \Gamma_D \cup \Gamma_S \).

\[
\begin{align*}
\Delta u & = 0 \text{ in } \Omega \\
u & = g \text{ on } \Gamma_D \\
\frac{\partial u}{\partial n} & = h \text{ on } \Gamma_N \\
u \leq g, \frac{\partial u}{\partial n} \leq h, \quad (u - g)(\frac{\partial u}{\partial n} - h) & = 0 \text{ on } \Gamma_S \\
g & = \begin{cases} 
0 & \text{on } \Gamma_D \\
-0.75 & \text{on } \Gamma_S 
\end{cases} \\
h & = n_x + n_y \text{ on } \Gamma_D \cup \Gamma_S 
\end{align*}
\]

Figure 3.5: The BEM (Laplace) model problem, geometry and boundary conditions

In Table 3.1 we list for the uniform \( h \) version the error estimate in the energy norm \( \\text{err}(u_{hp}) := \sqrt{2(J_{hp}(u_{hp}) - J(u))} \) (cf. Lemma 3.2), the value of the energy functional \( J_{hp}(u_{hp}) \) and the convergence rate \( \alpha \), as well as the numerical saturation value \( \bullet \). (The value of the energy functional for the exact solution \( J(u) = -0.32168035301 \) is obtained by extrapolation).

In Table 3.2 for the uniform \( h \) version we give error estimators obtained on one hand by the bubble subspace enrichment and on the other hand by the hat- and Haar-function subspace enrichment. These are the coarse grid error estimate \( \Theta_{hp} \), the Dirichlet error estimate \( \eta_{u, hp} \) and the Neumann error estimate \( \eta_{\varepsilon, hp} \) for the bubble case and the coarse grid error estimate \( \Theta_h \), the Dirichlet error estimate \( \eta_{u, h} \) and the Neumann error estimate \( \eta_{\varepsilon, h} \) for the hat/Haar case.

Table 3.3 contains the cpu times for computing the Galerkin matrix analytically, the Schur complement, for the solver (Polyak) and the computation of Dirichlet+Coarse grid and Neumann indicators. Note that the amount of work necessary to compute the
Dirichlet+Coarse grid indicator is large, but increases only as $O(N^2)$ and the computation of the Schur complement grows with $O(N^3)$. The numerical experiments have been done on a Pentium III (1GHz).

<table>
<thead>
<tr>
<th>dim $\sigma_{hp}$</th>
<th>$J_{hp}(u_{hp})$</th>
<th>err$(u_{hp})$</th>
<th>$\alpha$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-0.32156041260</td>
<td>0.0109372</td>
<td>0.501</td>
<td>0.501</td>
</tr>
<tr>
<td>128</td>
<td>-0.32164999048</td>
<td>0.0054813</td>
<td>0.997</td>
<td>0.500</td>
</tr>
<tr>
<td>256</td>
<td>-0.32167253600</td>
<td>0.0027385</td>
<td>1.001</td>
<td>0.499</td>
</tr>
<tr>
<td>512</td>
<td>-0.32167816977</td>
<td>0.0013658</td>
<td>1.004</td>
<td>0.501</td>
</tr>
<tr>
<td>1024</td>
<td>-0.32167956753</td>
<td>0.0006839</td>
<td>0.998</td>
<td>0.501</td>
</tr>
<tr>
<td>2048</td>
<td>-0.32167991801</td>
<td>0.0003425</td>
<td>0.998</td>
<td>0.998</td>
</tr>
</tbody>
</table>

Table 3.1: Errors, convergence rates and saturation numbers for the uniform $h$ version

For the adaptive version we have used the refinement parameters $\delta_1 = 0.8$ and $\delta_2 = 0.9$. In case of the $h$ version we refine the 20% elements with the largest indicators by bisection. In case of the $hp$ version we refine the 10% elements with the largest indicators.
Figure 3.8: Adaptively refined meshes, BEM approximations for $u$ and $\frac{\partial u}{\partial n}$ (after 0, 1, 2, 3 steps)

Figure 3.9: Adaptively generated meshes and polynomial degrees (after 1, 3, 5 steps)

Table 3.2: Bubble- and hat/Haar estimators for the uniform $h$ version

<table>
<thead>
<tr>
<th>dim $\sigma_{hp}$</th>
<th>$\Theta_{hp}$</th>
<th>$\eta_{h, hp}$</th>
<th>$\eta_{\varphi, hp}$</th>
<th>$\Theta_h$</th>
<th>$\eta_{h, h}$</th>
<th>$\eta_{\varphi, h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>0.0001724</td>
<td>0.0271207</td>
<td>0.0182034</td>
<td>.1388E-04</td>
<td>0.0229668</td>
<td>0.0167531</td>
</tr>
<tr>
<td>128</td>
<td>.4310E-04</td>
<td>0.0173710</td>
<td>0.0090475</td>
<td>.3131E-05</td>
<td>0.0114950</td>
<td>0.0083279</td>
</tr>
<tr>
<td>256</td>
<td>.1080E-04</td>
<td>0.0106595</td>
<td>0.0045235</td>
<td>.7857E-06</td>
<td>0.0057412</td>
<td>0.0041637</td>
</tr>
<tr>
<td>512</td>
<td>.2704E-05</td>
<td>0.0063359</td>
<td>0.0022611</td>
<td>.1948E-06</td>
<td>0.0028689</td>
<td>0.0020812</td>
</tr>
<tr>
<td>1024</td>
<td>.6758E-06</td>
<td>0.0036767</td>
<td>0.0011301</td>
<td>.4888E-07</td>
<td>0.0014346</td>
<td>0.0010402</td>
</tr>
<tr>
<td>2048</td>
<td>.1689E-06</td>
<td>0.0020944</td>
<td>0.0005650</td>
<td>.1219E-07</td>
<td>0.0007172</td>
<td>0.0005200</td>
</tr>
</tbody>
</table>

by increasing the polynomial degree by 1, whereas the next 10% elements will be refined by bisection. The remaining 80% of the elements will be kept unchanged.

We use a merged version of Algorithm 3.2, see Remark 3.6, i.e., we use the same partition for $\omega_h$ and $\gamma_h$, compute Dirichlet and Neumann indicators independently, and then we merge the refinement decisions into one mesh. The results are given in Table 3.4 for the $h$ version and in Table 3.5 for the $hp$ version both with bubble enrichment, whereas in Table 3.6 the results for hat/ Haar enrichment are given.

In Figure 3.8 we see the first meshes of the adaptive $h$ version with bubble enrichment. Note, that the meshes are correctly refined where the boundary conditions change. We recognize that the solution approaches the contact function $g$ only partly on $\Gamma_S$ and is
clearly not in contact on the remaining part. Figure 3.9 shows that our three-step algorithm (Algorithm 3.2) creates not only appropriate meshes but also increases the polynomial degrees as expected due to varying smoothness of the exact solution.

Table 3.3: Computing times (seconds) for the uniform $h$ version

<table>
<thead>
<tr>
<th>$\dim \sigma_h$</th>
<th>Gal</th>
<th>Schur</th>
<th>PK</th>
<th>$\eta_{h,h} + \Theta_h$</th>
<th>$\eta_{\sigma h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>0.04</td>
<td>0.01</td>
<td>0.00</td>
<td>0.13</td>
<td>0.06</td>
</tr>
<tr>
<td>128</td>
<td>0.15</td>
<td>0.06</td>
<td>0.02</td>
<td>0.61</td>
<td>0.23</td>
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<tr>
<td>256</td>
<td>0.56</td>
<td>0.79</td>
<td>0.12</td>
<td>3.54</td>
<td>0.91</td>
</tr>
<tr>
<td>512</td>
<td>2.31</td>
<td>7.73</td>
<td>1.30</td>
<td>20.51</td>
<td>3.62</td>
</tr>
<tr>
<td>1024</td>
<td>9.23</td>
<td>63.67</td>
<td>8.63</td>
<td>114.20</td>
<td>15.39</td>
</tr>
<tr>
<td>2048</td>
<td>42.15</td>
<td>607.38</td>
<td>48.93</td>
<td>970.21</td>
<td>62.24</td>
</tr>
</tbody>
</table>

Table 3.4: Error and (bubble-) estimators for the adaptive $h$ version

<table>
<thead>
<tr>
<th>$\dim \sigma_h$</th>
<th>$J_{hp}(u_{hp})$</th>
<th>$\operatorname{err}(u_{hp})$</th>
<th>$\Theta_{hp}$</th>
<th>$\eta_{h_{hp}}$</th>
<th>$\eta_{\sigma_{hp}}$</th>
<th>$\alpha$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-0.321556041260</td>
<td>0.0109372</td>
<td>0.0001724</td>
<td>0.0271207</td>
<td>0.0182034</td>
<td>0.481</td>
<td></td>
</tr>
<tr>
<td>78</td>
<td>-0.321652350160</td>
<td>0.0052617</td>
<td>.4363E-04</td>
<td>0.0173947</td>
<td>0.0093396</td>
<td>3.699</td>
<td>0.501</td>
</tr>
<tr>
<td>97</td>
<td>-0.32167307966</td>
<td>0.0026374</td>
<td>.1126E-04</td>
<td>0.0106966</td>
<td>0.0049359</td>
<td>3.168</td>
<td>0.553</td>
</tr>
<tr>
<td>124</td>
<td>-0.32167790520</td>
<td>0.0014595</td>
<td>.3088E-05</td>
<td>0.0063897</td>
<td>0.0027145</td>
<td>2.409</td>
<td>0.771</td>
</tr>
<tr>
<td>165</td>
<td>-0.32167777055</td>
<td>0.0011246</td>
<td>.9780E-06</td>
<td>0.0037381</td>
<td>0.0014667</td>
<td>0.913</td>
<td>0.523</td>
</tr>
<tr>
<td>221</td>
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<td>0.0005887</td>
<td>.2935E-06</td>
<td>0.0021402</td>
<td>0.0008680</td>
<td>2.215</td>
<td>0.573</td>
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<tr>
<td>300</td>
<td>-0.32167992164</td>
<td>0.0003371</td>
<td>.6975E-07</td>
<td>0.0012020</td>
<td>0.0004915</td>
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</tr>
<tr>
<td>411</td>
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<td>0.0002730</td>
<td>1.770</td>
<td>0.639</td>
</tr>
<tr>
<td>565</td>
<td>-0.32168002011</td>
<td>0.0001233</td>
<td>.3795E-07</td>
<td>0.0005778</td>
<td>0.0001605</td>
<td>1.410</td>
<td></td>
</tr>
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</table>

Table 3.5: Error and (bubble-) estimators for the adaptive $hp$ version

<table>
<thead>
<tr>
<th>$\dim \sigma_{hp}$</th>
<th>$J_{hp}(u_{hp})$</th>
<th>$\operatorname{err}(u_{hp})$</th>
<th>$\Theta_{hp}$</th>
<th>$\eta_{h_{hp}}$</th>
<th>$\eta_{\sigma_{hp}}$</th>
<th>$\alpha$</th>
<th>$\kappa$</th>
</tr>
</thead>
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<td>64</td>
<td>-0.32156028040</td>
<td>0.0109433</td>
<td>0.0001724</td>
<td>0.0271204</td>
<td>0.0182028</td>
<td></td>
<td></td>
</tr>
<tr>
<td>78</td>
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<td>0.0273632</td>
<td>0.0001328</td>
<td>0.0151396</td>
<td>0.0244663</td>
<td>0.667</td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>-0.32201296223</td>
<td>0.0182463</td>
<td>.4736E-04</td>
<td>0.0088917</td>
<td>0.0147943</td>
<td>1.700</td>
<td>0.628</td>
</tr>
<tr>
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<td>.1654E-04</td>
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<tr>
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<td>0.0029946</td>
<td>0.0052963</td>
<td>2.415</td>
<td>0.574</td>
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<td>.1895E-05</td>
<td>0.0017251</td>
<td>0.0030391</td>
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<td>.7919E-06</td>
<td>0.0010059</td>
<td>0.0017370</td>
<td>2.633</td>
<td>0.567</td>
</tr>
<tr>
<td>281</td>
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<td>0.0008054</td>
<td>0.0009621</td>
<td>3.173</td>
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</tr>
</tbody>
</table>
### Table 3.6: Error and (hat, Haar-) estimators for the adaptive $h$ version

<table>
<thead>
<tr>
<th>$\dim \sigma_h$</th>
<th>$J_{hp}(u_h)$</th>
<th>err($u_h$)</th>
<th>$\Theta_h$</th>
<th>$\eta_{u,h}$</th>
<th>$\eta_{\varphi,h}$</th>
<th>$\alpha$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>-0.32156041260</td>
<td>0.0109372</td>
<td>.1388E-04</td>
<td>0.0229668</td>
<td>0.0167531</td>
<td>0.473</td>
<td></td>
</tr>
<tr>
<td>78</td>
<td>-0.32165326626</td>
<td>0.0051739</td>
<td>.3125E-05</td>
<td>0.0116253</td>
<td>0.0087291</td>
<td>3.784</td>
<td>0.469</td>
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<tr>
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<td>0.0024272</td>
<td>.8131E-06</td>
<td>0.0050862</td>
<td>0.0046961</td>
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<tr>
<td>127</td>
<td>-0.32167871108</td>
<td>0.0011507</td>
<td>.2009E-06</td>
<td>0.0031707</td>
<td>0.0025881</td>
<td>2.879</td>
<td>0.710</td>
</tr>
<tr>
<td>170</td>
<td>-0.32167936720</td>
<td>0.0008174</td>
<td>.5124E-07</td>
<td>0.0018556</td>
<td>0.0014382</td>
<td>1.173</td>
<td>0.424</td>
</tr>
<tr>
<td>229</td>
<td>-0.32167991534</td>
<td>0.0003463</td>
<td>.1294E-07</td>
<td>0.0009816</td>
<td>0.0008362</td>
<td>2.883</td>
<td>0.694</td>
</tr>
<tr>
<td>312</td>
<td>-0.32167997759</td>
<td>0.0002402</td>
<td>.3335E-08</td>
<td>0.0005738</td>
<td>0.0004791</td>
<td>1.183</td>
<td>0.505</td>
</tr>
<tr>
<td>429</td>
<td>-0.3216802056</td>
<td>0.0001214</td>
<td>.9163E-09</td>
<td>0.0003119</td>
<td>0.0002677</td>
<td>2.143</td>
<td>0.782</td>
</tr>
<tr>
<td>591</td>
<td>-0.3216802630</td>
<td>.9488E-04</td>
<td>.1926E-09</td>
<td>0.0001876</td>
<td>0.0001535</td>
<td>0.769</td>
<td>0.588</td>
</tr>
<tr>
<td>823</td>
<td>-0.3216803219</td>
<td>.5580E-04</td>
<td>.4840E-10</td>
<td>0.0001038</td>
<td>.8649E-04</td>
<td>1.603</td>
<td>0.795</td>
</tr>
<tr>
<td>1144</td>
<td>-0.3216803333</td>
<td>.4438E-04</td>
<td>.1500E-10</td>
<td>.6395E-04</td>
<td>.5015E-04</td>
<td>0.695</td>
<td>0.793</td>
</tr>
<tr>
<td>1597</td>
<td>-0.3216803406</td>
<td>.3522E-04</td>
<td>.3057E-11</td>
<td>.4017E-04</td>
<td>.2961E-04</td>
<td>0.693</td>
<td></td>
</tr>
</tbody>
</table>

### Table 3.7: Computing times (seconds) for the adaptive $h$ version

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\dim \sigma_h$</th>
<th>$Gal$</th>
<th>$Schur$</th>
<th>$PK$</th>
<th>$\eta_{u,h} + \Theta_h$</th>
<th>$\eta_{\varphi,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>64</td>
<td>0.03</td>
<td>0.01</td>
<td>0.01</td>
<td>0.14</td>
<td>0.06</td>
</tr>
<tr>
<td>1</td>
<td>78</td>
<td>0.05</td>
<td>0.01</td>
<td>0.01</td>
<td>0.20</td>
<td>0.08</td>
</tr>
<tr>
<td>2</td>
<td>98</td>
<td>0.09</td>
<td>0.03</td>
<td>0.01</td>
<td>0.33</td>
<td>0.13</td>
</tr>
<tr>
<td>3</td>
<td>127</td>
<td>0.13</td>
<td>0.06</td>
<td>0.02</td>
<td>0.66</td>
<td>0.23</td>
</tr>
<tr>
<td>4</td>
<td>170</td>
<td>0.26</td>
<td>0.21</td>
<td>0.03</td>
<td>1.49</td>
<td>0.40</td>
</tr>
<tr>
<td>5</td>
<td>229</td>
<td>0.46</td>
<td>0.47</td>
<td>0.08</td>
<td>3.13</td>
<td>0.72</td>
</tr>
<tr>
<td>6</td>
<td>312</td>
<td>0.85</td>
<td>1.53</td>
<td>0.29</td>
<td>6.29</td>
<td>1.35</td>
</tr>
<tr>
<td>7</td>
<td>429</td>
<td>1.62</td>
<td>3.99</td>
<td>0.88</td>
<td>13.40</td>
<td>2.59</td>
</tr>
<tr>
<td>8</td>
<td>591</td>
<td>3.08</td>
<td>11.09</td>
<td>1.85</td>
<td>30.90</td>
<td>5.10</td>
</tr>
<tr>
<td>9</td>
<td>823</td>
<td>6.11</td>
<td>29.47</td>
<td>4.64</td>
<td>71.50</td>
<td>10.00</td>
</tr>
<tr>
<td>10</td>
<td>1144</td>
<td>11.98</td>
<td>90.69</td>
<td>10.50</td>
<td>152.62</td>
<td>19.00</td>
</tr>
<tr>
<td>11</td>
<td>1597</td>
<td>29.41</td>
<td>240.82</td>
<td>22.87</td>
<td>371.22</td>
<td>37.92</td>
</tr>
</tbody>
</table>
Example 3.2 In this example we investigate the same model problem as in Example 3.1 and Figure 3.5. We want to compare different solvers on an uniform mesh. In Table 3.8 we give the iteration numbers (using the relative change $\|x_{i+1} - x_i\|_\infty \leq \epsilon \|x_i\|_\infty$ as a stopping criterion with $\epsilon = 10^{-8}$) for the iterative solvers and in Table 3.9 we give the computing times for the Galerkin matrix, the Schur complement and the iterative solvers. Here, we solve the discrete system by the usual overrelaxation (SOR) algorithm with projection and damping parameter $\omega = 1.7$, by the Polyak algorithm without preconditioner, with the BPX preconditioner (see Theorem 3.11) and with the hierarchical basis preconditioner (see Theorem 3.9) and finally by the monotone multigrid algorithm (see Theorem 3.12).

We observe that the BPX-preconditioned Polyak algorithm is the best solver compared with the SOR-algorithm and the Polyak algorithm (PK) with hierarchical preconditioner and without preconditioner and the monotone multigrid methods. The solution time is clearly dominated by the computation of the Schur complement and we prefer schemes which eliminate the computation of the Schur complement, cf. Algorithm 3.3.

The principle behavior of the condition numbers of different preconditioners on the interval $[-1,1]$ for the hypersingular operator we can see in Figure 3.10. The behavior of the hypersingular operator is in principle the same as the behavior of the Steklov-Poincaré operator.

The numerical experiments have been done on a Sun Ultrasparc E450 (450MHz).

<table>
<thead>
<tr>
<th>N</th>
<th>SOR</th>
<th>PK</th>
<th>BPX-PK</th>
<th>HIER-PK</th>
<th>MMG</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>58</td>
<td>43</td>
<td>25</td>
<td>27</td>
<td>66</td>
</tr>
<tr>
<td>128</td>
<td>61</td>
<td>60</td>
<td>28</td>
<td>33</td>
<td>56</td>
</tr>
<tr>
<td>256</td>
<td>112</td>
<td>87</td>
<td>30</td>
<td>38</td>
<td>55</td>
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<td>217</td>
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<td>48</td>
</tr>
<tr>
<td>2048</td>
<td>791</td>
<td>241</td>
<td>33</td>
<td>51</td>
<td>45</td>
</tr>
</tbody>
</table>

Table 3.8: Iteration numbers for various solvers

<table>
<thead>
<tr>
<th>N</th>
<th>Gal</th>
<th>SCHUR</th>
<th>SOR</th>
<th>PK</th>
<th>BPX-PK</th>
<th>HIER-PK</th>
<th>MMG</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>0.0497</td>
<td>0.0184</td>
<td>0.0258</td>
<td>0.0066</td>
<td>0.0042</td>
<td>0.0044</td>
<td>0.0155</td>
</tr>
<tr>
<td>128</td>
<td>0.1927</td>
<td>0.1337</td>
<td>0.0192</td>
<td>0.0229</td>
<td>0.0119</td>
<td>0.0136</td>
<td>0.0505</td>
</tr>
<tr>
<td>256</td>
<td>0.7609</td>
<td>1.5169</td>
<td>0.1320</td>
<td>0.1178</td>
<td>0.0473</td>
<td>0.0562</td>
<td>0.1856</td>
</tr>
<tr>
<td>512</td>
<td>3.0457</td>
<td>18.424</td>
<td>1.6494</td>
<td>1.0135</td>
<td>0.2900</td>
<td>0.3747</td>
<td>0.8321</td>
</tr>
<tr>
<td>1024</td>
<td>12.284</td>
<td>237.48</td>
<td>18.377</td>
<td>8.0454</td>
<td>1.7871</td>
<td>2.4637</td>
<td>4.2040</td>
</tr>
<tr>
<td>2048</td>
<td>49.684</td>
<td>2137.3</td>
<td>215.55</td>
<td>68.040</td>
<td>10.421</td>
<td>15.314</td>
<td>25.120</td>
</tr>
</tbody>
</table>

Table 3.9: Computation times for various solvers
Figure 3.10: Condition numbers of BEM preconditioners
3.4.2 Examples for the Lamé equation

Example 3.3 Here, we investigate a half-disc with a radius=8 consisting of an elastic material with E-modul=2000, \( \sigma = 0.3 \) subject to the constant load \( L=1600 \) from above, i.e., \( \varphi = T(u) = (0, -L) \) on \( \Gamma_N \), pressed against a rigid obstacle. We approximate the circular boundary by a polygon using a quasi-uniform mesh, such that the number of elements on the straight line compared to the number of elements on the curved line have the ratio of 2/3. We are using piecewise linear, continuous splines for the displacement \( u \) and piecewise constant splines for the traction \( \varphi \).

The discrete system is solved by the usual overrelaxation (SOR) algorithm with projection and damping parameter \( \omega = 1.7 \) the monotone multigrid algorithm and the Polyak algorithm without preconditioner.

In Table 3.11 we give the iteration numbers (using the relative change \( |x_{i+1} - x_i|_{\infty} \leq \epsilon \|x_i\|_{\infty} \) as a stopping criterion with \( \epsilon = 10^{-8} \) and the computing times for the Galerkin matrix, the Schur complement and the iterative solvers. We observe that the Monotone Multigrid (MMG) is the far superior solver compared to the SOR-algorithm and the Polyak algorithm (PK) without preconditioner which is still better than SOR. The solution time is clearly dominated by the computation of the Schur complement.

In Table 3.10 we give the convergence rate of our discrete problem, making use of Lemma 2.4, a result analogous to Lemma 3.2 for the Lamé operator. We can estimate the error in the energy norm by \( \text{err}(u_N) := \sqrt{2(J_N(u_N) - J(u))} \), which we can use to determine the convergence rate. We have to note that this estimate is only valid approximately in our case, because one of the assumptions of Lemma 2.4 is not fulfilled, i.e., the convex subset used in the discrete problem is not a subset of the convex subset used in the continuous problem, due to the boundary approximation.

The exact potential is \( J(u) = 3214.187328 \) (obtained by extrapolation). We observe that we still get a good convergence rate, see Figure 3.11.

Figure 3.12 shows the original geometry and the deformed object.

The numerical experiments have been done on a Sun Ultrasparc E450 (450MHz).

<table>
<thead>
<tr>
<th>N</th>
<th>( J_N(u_N) )</th>
<th>err((u_N))</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>3239.700</td>
<td>5.0509690</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>3220.538</td>
<td>2.5199740</td>
<td>1.00</td>
</tr>
<tr>
<td>400</td>
<td>3215.751</td>
<td>1.2505401</td>
<td>1.01</td>
</tr>
<tr>
<td>800</td>
<td>3214.588</td>
<td>0.6326122</td>
<td>0.98</td>
</tr>
<tr>
<td>1600</td>
<td>3214.292</td>
<td>0.3235237</td>
<td>0.96</td>
</tr>
<tr>
<td>3200</td>
<td>3214.218</td>
<td>0.1754795</td>
<td>0.88</td>
</tr>
</tbody>
</table>

Table 3.10: Error in energy norm and convergence rates (Hertz contact problem)
Table 3.11: Iteration numbers and solution times (Hertz contact problem)

<table>
<thead>
<tr>
<th>N</th>
<th>SOR</th>
<th>MMG</th>
<th>PK</th>
<th>CPU-time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Gal</td>
</tr>
<tr>
<td>100</td>
<td>149</td>
<td>52</td>
<td>66</td>
<td>0.0798</td>
</tr>
<tr>
<td>200</td>
<td>297</td>
<td>58</td>
<td>105</td>
<td>0.3120</td>
</tr>
<tr>
<td>400</td>
<td>651</td>
<td>56</td>
<td>225</td>
<td>1.2335</td>
</tr>
<tr>
<td>800</td>
<td>1354</td>
<td>62</td>
<td>300</td>
<td>4.9729</td>
</tr>
<tr>
<td>1600</td>
<td>2734</td>
<td>74</td>
<td>633</td>
<td>20.039</td>
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<tr>
<td>3200</td>
<td>5424</td>
<td>86</td>
<td>1256</td>
<td>80.310</td>
</tr>
</tbody>
</table>

Figure 3.11: Convergence of the Hertz contact problem
Figure 3.12: Hertz contact problem
Chapter 4

FEM-BEM coupling with Signorini contact

In this chapter we will investigate the interface problem with Signorini contact conditions originally described in [23]. From this work we will use the variational formulation as a FEM-BEM coupling problem. In Section 4.1 we will recall briefly the results on uniqueness and existence of a continuous solution and the convergence results of the discretized variational formulation from [23]. Here we will also give a convergence proof of the $p$ version. Additionally we will create the framework for the following sections. In Section 4.2 we will prove the properties of an hierarchical a posteriori error estimator. In Section 4.3 we will give some results on solving this system. Especially for the linear case we will prove the efficiency of multigrid and BPX preconditioners for this coupling system. Finally, in Section 4.4 we will present some numerical examples underlining the theory.

4.1 Formulation of the FEM-BEM coupling

4.1.1 Variational formulation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\Gamma$. Let $\Gamma = \Gamma_t \cup \Gamma_s$ where $\Gamma_t$ and $\Gamma_s$ are nonempty, disjoint and open in $\Gamma$. In the interior part $\Omega$ we consider a nonlinear partial differential equation, whereas in the exterior part $\Omega_c = \mathbb{R}^n \setminus \Omega$ we consider the Laplace equation.

$$-
\text{div}(\varrho(|\nabla u|) \cdot \nabla u) = f \text{ in } \Omega$$

$$-\Delta u = 0 \text{ in } \Omega_c = \mathbb{R}^n \setminus \Omega$$

with the radiation condition as $|x| \to \infty$

$$u(x) = \begin{cases} 
  a + o(1) & \text{if } n = 2, \\
  O(|x|^{2-n}) & \text{if } n \geq 3,
\end{cases}$$

(4.3)

where $a \in \mathbb{R}$ is constant for any $u$ but varying with $u$.

Here $\varrho : [0, \infty) \to [0, \infty)$ is a $C^1[0, \infty)$ function with $t \cdot \varrho(t)$ being monotonously increasing with $t$, $\varrho(t) \leq \varrho_0$, $(t \cdot \varrho(t))^\prime \leq \varrho_1$ and let $\varrho(t) + t \cdot \min\{0, \varrho(t)\} \geq \alpha > 0$.

Writing $u_1 := u_{|\Omega}$ and $u_2 := u_{|\Omega_c}$, the tractions on $\Gamma$ are given by $\varrho(|u_1|) \frac{\partial u_1}{\partial n}$ and $-\frac{\partial u_2}{\partial n}$ with normal vector $n$ pointing into $\Omega_c$.  

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We consider transmission conditions on $\Gamma_t$, 

$$
u_1|_{\Gamma_t} - u_2|_{\Gamma_t} = u_0|_{\Gamma_t} \quad \text{and} \quad g(|\nabla u_1|) \left( \frac{\partial u_1}{\partial n} \right)|_{\Gamma_t} - \frac{\partial u_2}{\partial n} \right)|_{\Gamma_t} = t_0|_{\Gamma_t}, \tag{4.4}$$

and Signorini conditions on $\Gamma_s$,

$$u_1|_{\Gamma_s} - u_2|_{\Gamma_s} \leq u_0|_{\Gamma_s} \quad \text{and} \quad \frac{\partial u_1}{\partial n} \right)|_{\Gamma_s} = \frac{\partial u_2}{\partial n} \right)|_{\Gamma_s} \leq 0.$$ \tag{4.5}

In order to derive a variational formulation of (4.1)–(4.5) we assume that the given data satisfy $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, $t_0 \in H^{-1/2}(\Gamma)$ and we look for $u_1 \in H^1(\Omega)$ and $u_2 \in H^{1}_{loc}(\Omega_c)$.

Now, we introduce the following operators and forms. Let the function $g$ be given by

$$g : [0, \infty) \to [0, \infty), \quad t \mapsto g(t) = \int_0^t s \cdot g(s) \, ds.$$

From the assumptions on $g(t)$ it follows that $0 \leq g(t) \leq \frac{1}{2} \varrho_0 \cdot t^2$ and hence

$$G(u) := \int_{\Omega} g(|\nabla u|) \, dx$$

is finite for any $u \in H^1(\Omega)$. The Fréchet derivative of $G$ is

$$DG(u; v) = \int_{\Omega} \frac{\partial u_1}{\partial n}(\nabla u)^T \cdot \nabla v \, dx \quad \forall u, v \in H^1(\Omega) \tag{4.6}$$

and $DG$ is uniformly monotone with respect to the semi-norm $| \cdot |_{H^1(\Omega)} = \| \nabla \cdot \|_{L^2(\Omega)}$ in $H^1(\Omega)$, i.e., there exists a constant $\gamma > 0$ such that

$$\gamma |u - v|_{H^1(\Omega)}^2 \leq DG(u; u - v) - DG(v; u - v) \quad \forall u, v \in H^1(\Omega) \tag{4.7}$$

(see [23, Proposition 2.1]).

The linear functional $L : H^1(\Omega) \times H^{1/2}(\Gamma) \to \mathbb{R}$ is given by

$$L(u, v) := \int_{\Omega} f \cdot u \, dx + \langle t_0, v \rangle.$$ 

We may define the Steklov-Poincaré operator for the exterior problem

$$S := \frac{1}{2} (W + (K' - I)V^{-1}(K - I)) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma). \tag{4.8}$$

**Remark 4.1** The Steklov-Poincaré operator for the exterior problem differs from the Steklov-Poincaré operator for the interior problem defined in (3.5) by the sign in $K - I$ instead of $K + I$ for the interior problem. As a consequence the Steklov-Poincaré operator for the exterior problem is positive definite on $H^{1/2}(\Gamma)$ and not positive semi-definite as for the interior problem.
**Definition 4.1** Let $E := H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)$ where $\tilde{H}^{1/2}(\Gamma_s) := \{w \in H^{1/2}(\Gamma) : \text{supp } w \subseteq \Gamma_s\}$. Define $\Psi : E \to \mathbb{R}$ by

$$
\Psi(u, v) := \int_\Omega g(|\nabla u|) \, dx + \frac{1}{2} \langle S(u|_\Gamma + v), u|_\Gamma + v \rangle - \lambda(u, v),
$$

where $\lambda \in E^*$ is given by

$$
\lambda(u, v) := L(u, u|_\Gamma + v) + \langle Su_0, u|_\Gamma + v \rangle
$$

for any $(u, v) \in E$. Let

$$
D := \{(u, v) \in E : v \geq 0 \text{ a.e. on } \Gamma_s \text{ and, if } n = 2, \langle S1, u|_\Gamma + v - u_0 \rangle = 0\}.
$$

Then the problem $(P)$ consists in finding a minimizer $(\hat{u}, \hat{v})$ in $D$ of $\Psi$, i.e., $(\hat{u}, \hat{v}) \in D$ such that

$$
\Psi(\hat{u}, \hat{v}) = \inf_{(u, v) \in D} \Psi(u, v).
$$

**Remark 4.2** It has been shown in [23, Proposition (v)] that $L(1, 1) = 0$ is necessary for uniqueness of the solution for $n = 2$.

**Theorem 4.1** [23, Theorem 4.2] If $n = 2$, then let $L(1, 1) = 0$. There exists exactly one solution $(\hat{u}, \hat{v}) \in D$ of problem $(P)$, which is the variational solution of the transmission problem. Moreover, $(\hat{u}, \hat{v}) \in D$ is the unique solution of the following variational inequality:

Find $(\hat{u}, \hat{v}) \in D$ such that

$$
\mathcal{A}(\hat{u}, \hat{v}; u - \hat{u}, v - \hat{v}) \geq \lambda(u - \hat{u}, v - \hat{v}) \tag{4.9}
$$

for all $(u, v) \in D$, where $\mathcal{A} : E \to E^*$ is defined via

$$
\mathcal{A}(u, v; r, s) := DG(u, r) + \langle S(u|_\Gamma + v), r|_\Gamma + s \rangle. \tag{4.10}
$$

**Remark 4.3** For $n = 2$ let $(\hat{U}, \hat{V}) \in \{(u, v) \in E : v \geq 0 \text{ a.e. on } \Gamma_s\}$ be the solution of

$$
\mathcal{A}(\hat{U}, \hat{V}; U - \hat{U}, V - \hat{V}) \geq \lambda(U - \hat{U}, V - \hat{V}) \quad \forall (U, V) \in \{(u, v) \in E : v \geq 0 \text{ a.e. on } \Gamma_s\}. \tag{4.11}
$$

Let $c \in \mathbb{R}$ be arbitrary. Choosing $(U, V) = (\hat{U} + c, \hat{V})$ in $(4.11)$ we obtain

$$
\mathcal{A}(\hat{U}, \hat{V}; c, 0) \geq \lambda(c, 0) \quad \forall c \in \mathbb{R}
$$

which gives

$$
\langle S(\hat{U}|_\Gamma + \hat{V}), c|_\Gamma \rangle \geq L(c, c) + \langle Su_0, c|_\Gamma \rangle \quad \forall c \in \mathbb{R}.
$$

Due to [23, Proposition (v)] we have $L(c, c) = cL(1, 1) = 0$ and therefore

$$
\langle S(\hat{U}|_\Gamma + \hat{V} - u_0), c|_\Gamma \rangle \geq 0 \quad \forall c \in \mathbb{R},
$$

i.e., we have $(\hat{U}, \hat{V}) \in D$ and $(\hat{u}, \hat{v}) = (\hat{U}, \hat{V})$. As a consequence, we see that it is sufficient to consider the variational formulation $(4.9)$ on $\{(u, v) \in E : v \geq 0 \text{ a.e. on } \Gamma_s\}$ instead of $D$. The corresponding discretization is much simpler, due to the missing restriction given by $\langle S1, u|_\Gamma + v - u_0 \rangle = 0$. 

**Definition 4.2** Let the bilinear form \( B \) be defined by
\[
B(u, \phi; v, \psi) := \langle Wu, v \rangle + \langle (I - K') \phi, v \rangle - \langle (I - K) u, \psi \rangle + \langle V \phi, \psi \rangle
\]  
(4.12)
for all \( u, v \in H^{1/2}(\Gamma) \) and \( \phi, \psi \in H^{-1/2}(\Gamma) \).

Then the problem \((P_1)\) reads:

Find \((\tilde{u}, \tilde{v}, \tilde{\phi}) \in D \times H^{-1/2}(\Gamma)\) such that
\[
DG(\tilde{u}; u - \tilde{u}) + \frac{1}{2} B(\tilde{u}|_{\Gamma} + \tilde{v} - u_0, \phi - \tilde{\phi}; u|_{\Gamma} + v - (\tilde{u}|_{\Gamma} + \tilde{v}), \phi - \tilde{\phi}) \geq \int_{\Omega} f \cdot (u - \tilde{u}) \, dx + \langle t_0, u|_{\Gamma} + v - (\tilde{u}|_{\Gamma} + \tilde{v}) \rangle
\]  
(4.13)
for all \((u, v, \phi) \in D \times H^{-1/2}(\Gamma)\).

**Lemma 4.1** The problems \((P)\) and \((P_1)\) are equivalent.

**Proof.** Isolating the part containing \( \phi - \tilde{\phi} \) from the variational inequality (4.13) we have
\[
0 = -\langle (I - K)(\tilde{u}|_{\Gamma} + \tilde{v} - u_0), \phi - \tilde{\phi} \rangle + \langle V \tilde{\phi}, \phi - \tilde{\phi} \rangle \quad \forall \phi \in H^{-1/2}(\Gamma).
\]
Therefore we have
\[
\tilde{\phi} = V^{-1}(I - K)(\tilde{u}|_{\Gamma} + \tilde{v} - u_0).
\]
Inserting this into (4.13) we obtain the Steklov-Poincaré operator and the linear functional \( \lambda(\cdot, \cdot) \).

### 4.1.2 Numerical approximation

In this section we consider the numerical approximation for problem \((P)\) by a Galerkin projection using finite elements in \( \Omega \) and boundary elements on \( \Gamma \). Let
\[
(H^1 \times \tilde{H}^{1/2} \times H^{-1/2})_{h \in I}
\]
be a family of finite-dimensional subspaces of \( H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \times H^{-1/2}(\Gamma) \), where \( I \subseteq (0, \infty) \) with 0 \( \in \bar{I} \). Note that \( H^1_h, \tilde{H}^{1/2}_h \) and \( H^{-1/2}_h \) correspond to triangulations of \( \Omega, \Gamma_s \) and \( \Gamma \). For simplicity we assume that \( \Gamma_1 \) and \( \Gamma_s \) are polygonal (i.e., piecewise straight lines) for \( n = 2 \) or piecewise hyperplanes for \( n \geq 3 \).

We assume the approximation property, i.e., for any \((u, v, \phi) \in H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \times H^{-1/2}(\Gamma)\) we have
\[
0 = \lim_{I_{3h} \to 0} \text{dist}_{H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \times H^{-1/2}(\Gamma)}((u, v, \phi), H^1_h \times \tilde{H}^{1/2}_h \times H^{-1/2}_h).
\]  
(4.14)
In addition, we assume for all \( h_1, h_2 \in I \) that
\[
h_1 < h_2 \quad \Rightarrow \quad H^{-1/2}_{h_2} \subseteq H^{-1/2}_{h_1},
\]  
(4.15)
which means that we consider a hierarchical discretization on the boundary. For \( h \in I \) let
\[
i_h : H^1_h \hookrightarrow H^1(\Omega),
\]
\[
j_h : \tilde{H}^{1/2}_h \hookrightarrow \tilde{H}^{1/2}(\Gamma),
\]
\[
k_h : H^{-1/2}_h \hookrightarrow H^{-1/2}(\Gamma)
\]
CHAPTER 4. FEM-BEM COUPLING WITH SIGNORINI CONTACT

denote the canonical imbeddings with their duals $i_h^*, j_h$, and $k_h^*$ and let $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ be the trace operator with dual $\gamma^*$.

Let $\lambda_h \in (H^1_h \times \tilde{H}^{1/2}_h)^*$ and let $D_h$ be a convex, closed non-void subset of $H^1_h(\Omega) \times \tilde{H}^{1/2}_h(\Gamma)$.

In order to approximate $S$ we define

$$S_h := i_{h}^{*} \gamma^* W \gamma i_h + i_{h}^{*} \gamma^* (I - K') k_h (k_h^* V k_h)^{-1} k_h^* (I - K) \gamma i_h.$$ 

We recall two essential properties of $S_h$.

**Lemma 4.2** [19] There exist $c_0 > 0$ and $h_0 \in I$ such that for any $h \in I$ with $h < h_0$ and any $u_h \in H^1_h$ there holds

$$\langle S_h u_h, u_h \rangle \geq c_0 \| \gamma i_h u_h \|_{H^{1/2}(\Gamma)}^2.$$  

**Lemma 4.3** [30, Equation (3.15)], [19, Lemma 9] There exists a constant $c > 0$ such that for all $h \in I$ and for all $u_h \in H^1_h$ there holds

$$\| S_h u_h - i_{h}^{*} \gamma^* S \gamma u_h \|_{H^{-1/2}(\Gamma)} \leq c \text{dist}_{H^{-1/2}(\Gamma)}(V^{-1} (I - K) \gamma u_h; H^{-1/2}_h).$$  

**Definition 4.3** The problem $(P_h)$ is the following: Find $(\hat{u}_h, \hat{v}_h) \in D_h$ such that

$$A_h(\hat{u}_h, \hat{v}_h; u_h - \hat{u}_h, v_h - \hat{v}_h) \geq \lambda_h(u_h - \hat{u}_h, v_h - \hat{v}_h) \quad \text{for all } (u_h, v_h) \in D_h,$$

where $A_h : H^1_h \times \tilde{H}^{1/2}_h \to (H^1_h \times \tilde{H}^{1/2}_h)^*$ is defined via

$$A_h(u_h, v_h; r_h, s_h) := DG(u_h, v_h) + \langle S_h(u_h|_\Gamma + v_h), r_h|_\Gamma + s_h \rangle.$$  

and $\lambda_h : H^1_h \times \tilde{H}^{1/2}_h \to \mathbb{R}$ is defined via

$$\lambda_h(u_h, v_h) := L(u_h, u_h|_\Gamma + v_h) + (i_{h}^{*} \gamma^* (W + (I - K') k_h (k_h^* V k_h)^{-1} k_h^* (I - K)) u_h, u_h|_\Gamma + v_h).$$

**Lemma 4.4** There exists a constant $c > 0$ such that for all $(u_h, v_h), (r_h, s_h) \in H^1_h \times \tilde{H}^{1/2}_h$, there holds

$$c \cdot \| (u_h - r_h, v_h - s_h) \|_{H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma)}^2 \leq A_h(u_h, v_h; u_h - r_h, v_h - s_h) - A_h(r_h, s_h; u_h - r_h, v_h - s_h).$$

**Proof.** Due to [23, Lemma 4.1] there holds for a constant $c' > 0$

$$c' \cdot \| (u_h - r_h, v_h - s_h) \|_{H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma)}^2 \leq DG(u_h; u_h - r_h) - DG(r_h; u_h - r_h) + \langle S(u_h|_\Gamma + v_h - (r_h|_\Gamma + s_h)), u_h|_\Gamma + v_h - (r_h|_\Gamma + s_h) \rangle.$$  

$DG$ is uniformly monotone with respect to the $H^1(\Omega)$ semi-norm, therefore we have $DG(u_h; u_h - r_h) - DG(r_h; u_h - r_h) \geq 0$. From the continuity of $S$ in the $H^{1/2}(\Gamma)$ norm and Lemma 4.2 we obtain

$$\langle S(u_h|_\Gamma + v_h - (r_h|_\Gamma + s_h)), u_h|_\Gamma + v_h - (r_h|_\Gamma + s_h) \rangle \leq c'' \| u_h|_\Gamma + v_h - (r_h|_\Gamma + s_h) \|_{H^{1/2}(\Gamma)}^2 \leq c'' \cdot c_0 \langle S_h(u_h|_\Gamma + v_h - (r_h|_\Gamma + s_h)), u_h|_\Gamma + v_h - (r_h|_\Gamma + s_h) \rangle.$$
This concludes the proof. □

In [23, Theorem 6.1] an abstract error estimate for nonlinear variational inequalities is given. Its application in [23, Theorem 7.1] to a discretization assumes that the Steklov-Poincaré operator appearing in the linear functional \( \lambda(\cdot, \cdot) \) can be consistently discretized without a Schur complement. In general, a numerical implementation can only be realized using the Schur complement approximation of the Steklov-Poincaré operator \( A(\cdot, \cdot, \cdot) \) and in \( \lambda(\cdot, \cdot) \). Therefore we have to reinvestigate the application of the abstract error estimate to our problem, taking into account the approximation of \( \lambda(\cdot, \cdot) \).

**Theorem 4.2** Let \((\hat{u}, \hat{v})\) solve problem \((P)\), and for any \( h \in I \) with \( h < h_0 \) (\( h_0 \) from Lemma 4.2) let \((\hat{u}_h, \hat{v}_h)\) be the solution of the discrete problem \((P_h)\). Assume that there exists \( x_0 \in E := H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \) with

\[
x_0 \in D \cap \bigcap_{h < h_0} D_h.
\]

Let \( F \subseteq E \subseteq G \) be Banach spaces with continuous embeddings such that \((\hat{u}, \hat{v}) \in F \) and \( A(\hat{u}, \hat{v}; \cdot, \cdot) - \lambda \in G^* \). Let \( E_h := H^1_h \times \tilde{H}^{1/2} \). Then there exist some constants \( c_1, c_2, \gamma, c_B > 0 \) (independent of \( h \)) such that

\[
\gamma \|(\hat{u} - \hat{u}_h)\|_{H^1(\Omega)}^2 + \frac{c_B}{4} \|(\hat{u} - \hat{u}_h)\|_{H^1(\Gamma)}^2 \leq c_1 \cdot \text{dist}_{H^{-1/2}(\Gamma)}(V^{-1}(I - K)(\hat{u}_\Gamma + \hat{v}), H^{-1/2}(\Gamma))^2 + c_2 \cdot \text{dist}_{H^{-1/2}(\Gamma)}(V^{-1}(I - K)u_0, H^{-1/2}(\Gamma))^2
\]

\[
+ 2\|A(\hat{u}, \hat{v}; \cdot, \cdot) - \lambda\|_{G^*} \inf_{(u, v) \in E} \|((\hat{u}_h - u, \hat{v}_h - v))\|_G
\]

\[
+ \frac{1}{2}\|A(\hat{u}, \hat{v}; \cdot, \cdot) - \lambda\|_{G^*} \|(\hat{u} - u_h, \hat{v} - v_h)\|_G^2.
\]

**Proof.** In the following we use the notation introduced in [23, Theorem 6.1]. We define

\[
A(u, v)(w, \eta) := DG(u; w), \quad B(u, v)(w, \eta) := \langle S(u_\Gamma + v), w_\Gamma + \eta \rangle
\]

for \((u, v), (w, \eta) \in E\), and

\[
A_h(u_h, v_h)(w_h, \eta_h) := DG(u_h; w_h), \quad B_h(u_h, v_h)(w_h, \eta_h) := \langle S_h(u_h_\Gamma + v_h), w_h_\Gamma + \eta_h \rangle
\]

for \((u_h, v_h), (w_h, \eta_h) \in E_h\), and

\[
\|(u, v)\|_A := \|u\|_{H^1(\Omega)} \quad \text{and} \quad \|(u, v)\|_B := \|u_\Gamma + v\|_{H^{1/2}(\Gamma)} \quad \forall (u, v) \in E,
\]

so that, as seen in [23, Lemma 4.1], \( \| \cdot \|_E \) and \( \| \cdot \|_A + \| \cdot \|_B \) are equivalent. Due to

\[
\gamma |u - v|_{H^1(\Omega)}^2 \leq DG(u; u - v) - DG(v; u - v) \quad \forall u, v \in H^1(\Omega)
\]

the functional \( \alpha(\cdot) \) is given by \( \alpha(s) = \gamma s^2 \), and, consequently, the Fenchel conjugate is

\[
\alpha^*(t) := \sup_{s \geq 0} \{s \cdot t - \alpha(s)\} = \frac{1}{4\gamma} t^2.
\]
Applying now [23, Theorem 6.1] we obtain
\[
\gamma |\hat{u} - \hat{u}_h|^2_{H^1(\Omega)} + \frac{c_B}{4} \| (\hat{u} - \hat{u}_h)|\Gamma + \hat{v} - \hat{v}_h|^2_{H^{1/2}(\Gamma)} \\
\leq c_1 \cdot \text{dist}_{H^{-1/2}(\Gamma)}(V^{-1}(I - K)(\hat{u}_\Gamma + \hat{v}), H^{-1/2}_h)^2 + c_1 \cdot \| (\hat{u}_h \times j_h)^\ast \lambda - \lambda_h \|^2_{E_h^\ast} \\
+ 2\| \mathcal{A}(\hat{u}, \hat{v}; \cdot, \cdot) - \lambda \| G^\ast \| ([\hat{u}_h - u, \hat{v}_h - v]) \|_G + \inf_{(u_h, v_h) \in D_h \cap F} \frac{1}{2\gamma} \left( \frac{1}{2\gamma} \| (\hat{u} - u_h, \hat{v} - v_h) \|_F + \| (\hat{u}_h \times j_h)^\ast \lambda - \lambda_h \|^2_{E_h^\ast} \right) \]
\[+ \left( \frac{1}{2\gamma} \| (\hat{u} - u_h, \hat{v} - v_h) \|_F + \| (\hat{u}_h - u_h, \hat{v}_h - v_h) \|_F^2 \right) \]
\[+ 2\| \mathcal{A}(\hat{u}, \hat{v}; \cdot, \cdot) - \lambda \| G^\ast \| ([\hat{u}_h - u_h, \hat{v}_h - v_h]) \|_G \right) \].
\]

Analogously to [18, Lemma 2.4] we have
\[
\| (\hat{u}_h \times j_h)^\ast \lambda - \lambda_h \|^2_{E_h^\ast} \leq c \text{dist}_{H^{-1/2}(\Gamma)}(V^{-1}(I - K)u_0, H^{-1/2}_h).
\]

\[ \square \]

4.1.3 Numerical approximation with the $h$ version

For a family of mesh parameters $h$ we have nested regular quasi-uniform meshes $(T_h)_h$ consisting of triangles or quadrilaterals. Then, let $H^1_h$ denote the related continuous and piecewise affine trial functions on the triangulation $T_h$. The mesh on $\Omega$ induces a mesh on the boundary, so that we may consider $H^{-1/2}_h$ as the piecewise constant trial functions. Assuming that the partition of the boundary leads also to a partition of $\Gamma$, $\hat{H}^{-1/2}_h$ is then the subspace of continuous and piecewise linear functions on the partition of $\Gamma$. Then we have $H^1_h \times \hat{H}^{-1/2}_h \subset H^1(\Omega) \times H^{-1/2}(\Gamma_s) \times H^{-1/2}(\Gamma)$. Now, $D_h$ is given by
\[
D_h := \{ (u_h, v_h) \in H^1_h \times \hat{H}^{-1/2}_h : v(x_i) \geq 0, \forall x_i \text{ node of the partition of } \Gamma_s, \\
\text{and for } n = 2, \langle S1, u_h|\Gamma + v_h - u_0 \rangle = 0 \}.
\]

(4.23)

Note that $v_h \geq 0$, once the nodal values of $v_h$ are $\geq 0$. Therefore we have $D_h \subset D$.

**Theorem 4.3** Let $(\hat{u}_h, \hat{v}_h) \in D_h$ be the solution of
\[
\mathcal{A}_h(\hat{u}_h, \hat{v}_h; u_h - \hat{u}_h, v_h - \hat{v}_h) \geq \lambda_h(u_h - \hat{u}_h, v_h - \hat{v}_h) \quad \forall (u_h, v_h) \in D_h.
\]

Then there holds
\[
\lim_{h \to 0} \| (\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h) \|_{H^1(\Omega) \times \hat{H}^{-1/2}(\Gamma_s)} = 0.
\]

(4.25)

**Proof.** Applying now Theorem 4.2 and identifying the spaces $E = F = G = H^1(\Omega) \times \hat{H}^{-1/2}(\Gamma_s)$, we obtain
\[
c \cdot \| (\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h) \|^2_E \leq \text{dist}_{H^{-1/2}(\Gamma)(V^{-1}(I - K)(\hat{u}_\Gamma + \hat{v}), H^{-1/2}_h)^2} + \text{dist}_{H^{-1/2}(\Gamma)(V^{-1}(I - K)u_0, H^{-1/2}_h)^2} \]
\[+ \inf_{(u_h, v_h) \in D_h} \left\{ \| (\hat{u} - u_h, \hat{v} - v_h) \|^2_E + 2\| \mathcal{A}(\hat{u}, \hat{v}; \cdot, \cdot) - \lambda \| E^\ast \| ([\hat{u}_h - u_h, \hat{v}_h - v_h]) \|_E \right\}.
\]

Due to the approximation property of the subspaces involved the upper bound of the error tends to zero. \( \square \)
4.1.4 Numerical approximation with the \(hp\) version

In this section we will treat a more general approach than the \(h\) version discussed before. We replace the index \(h\) used before by the index \(hp\), indicating the increasing polynomial degrees of the quasi-uniform \(hp\) version. The general approximation properties formulated in Section 4.1.2 are still valid.

Let \(T_h\) be a regular quasi-uniform mesh consisting of triangles and quadrilaterals which will be kept fixed in the \(p\) version. Let \(p \geq 1\). Then let \(H^1_{hp}\) denote the space of continuous trial functions given by piecewise polynomials of degree \(p\) in each variable on quadrilaterals and of total degree \(p\) on triangles. In case of quadrilaterals a suitable basis is given by the tensor product of antiderivatives of Legendre polynomials up to degree \(p\). On the partition of \(\Gamma\) given by \(T_h|\Gamma\) we define \(H^{-1/2}_{hp}\) to be piecewise polynomials of degree \(\leq p-1\), a suitable basis are the Legendre polynomials up to degree \(p-1\).

Now we proceed similar to Section 3.1.2: On the interval \([-1,1]\) we choose \(N+1\) Gauss-Lobatto points, i.e., the points \(\xi_j^{N+1}, 0 \leq j \leq N\), that are the zeros of \((1-\xi^2)L_N'(\xi)\), where \(L_N\) denotes the Legendre polynomial of degree \(N\). For these points it is known (cf. [10, Prop. 2.2, (2.3)]) that there exist positive weight factors \(\varrho_j^{N+1} := \frac{1}{N(N+1)L_N'(\xi_j^{N+1})}\) such that

\[
\forall \phi \in \mathcal{P}_{2N-1}([-1,1]) : \sum_{j=0}^N \varrho_j^{N+1} \phi(\xi_j) \varrho_j^{N+1} = \int_{-1}^1 \phi(\xi) \, d\xi. \tag{4.26}
\]

By an affine transformation we define the set of Gauss-Lobatto points \(G_{e, hp}\) on each element \(e\) of the partition on \(\Gamma_s\), induced by \(T_h\), corresponding to the polynomial degree \(p\), i.e., we use the Gauss-Lobatto formula with \(p+1\) nodes (see [65]) and we set

\[
G_{hp} := \bigcup_{e\in T_h|\Gamma_s} G_{e, hp}.
\]

On the partition of \(\Gamma_s\) given by \(T_h|\Gamma_s\), we define \(\tilde{H}^{1/2}_hp \subset \tilde{H}^{1/2}(\Gamma_s)\) to be the space of continuous polynomials of degree \(\leq p\), which vanish at the endpoints of \(\Gamma_s\). A suitable basis of \(\tilde{H}^{1/2}_hp\) are the Lagrange interpolation polynomials to the set of Gauss-Lobatto points on each element, because we later have to evaluate the values of the solution on the set \(G_{hp}\).

Then we have \(H^1_{hp} \times \tilde{H}^{1/2}_hp \times H^{-1/2}_{hp} \subset H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \times H^{-1/2}(\Gamma)\). Now, \(D_{hp}\) is given by

\[
D_{hp} := \left\{ (u_{hp}, v_{hp}) \in H^1_{hp} \times \tilde{H}^{1/2}_hp : v_{hp}(x_i) \geq 0, \forall x_i \in G_{hp}
\right. \]

and if \(n = 2\), \(\langle S, u_{hp}|_{\Gamma} + v_{hp} - u_0 \rangle = 0\). \tag{4.27}

\(D_{hp}\) is a convex, closed subset of \(H^1_{hp} \times \tilde{H}^{1/2}_hp\). Note, for \(p \geq 2\), we have \(D_{hp} \subsetneq \tilde{D}\).

**Theorem 4.4** Let \(\varrho \equiv 1\) and let \((\hat{u}_{hp}, \hat{v}_{hp}) \in D_{hp}\) be the solution of

\[
\mathcal{A}_{hp}(\hat{u}_{hp}, \hat{v}_{hp}; u_{hp} - \hat{u}_{hp}, v_{hp} - \hat{v}_{hp}) \geq \lambda_{hp}(u_{hp} - \hat{u}_{hp}, v_{hp} - \hat{v}_{hp}) \quad \forall (u_{hp}, v_{hp}) \in D_{hp}. \tag{4.28}
\]

Then there hold

\[
\lim_{p \to \infty} \|(\hat{u} - \hat{u}_{hp}, \hat{v} - \hat{v}_{hp})\|_{H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)} = 0, \quad \text{if \(h\) fixed} \tag{4.29}
\]

and

\[
\lim_{h \to \infty} \|(\hat{u} - \hat{u}_{hp}, \hat{v} - \hat{v}_{hp})\|_{H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)} = 0, \quad \text{if \(p\) fixed} \tag{4.30}
\]
Proof. Since \( \rho \equiv 1 \) the form \( A(\cdot,\cdot,\cdot) \) is bilinear and positive definite (see [23, Lemma 4.1]). We define the following auxiliary problem:

Let \( (\bar{u}_{hp}, \bar{v}_{hp}) \in D_{hp} \) solve

\[
A(\bar{u}_{hp}, \bar{v}_{hp}; u_{hp} - \bar{u}_{hp}, v_{hp} - \bar{v}_{hp}) \geq \lambda(u_{hp} - \bar{u}_{hp}, v_{hp} - \bar{v}_{hp}) \quad \forall (u_{hp}, v_{hp}) \in D_{hp}.
\]

(4.31)

Due to Lemma 4.5 and Lemma 4.6 the assumptions of [39, Theorem I.5.2] hold and we obtain

\[
\lim_{p \to \infty} \| (\hat{u} - \bar{u}_{hp}, \hat{v} - \bar{v}_{hp}) \|_{H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)} = 0, \text{ if } h \text{ fixed}
\]

and

\[
\lim_{h \to 0} \| (\hat{u} - \bar{u}_{hp}, \hat{v} - \bar{v}_{hp}) \|_{H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)} = 0, \text{ if } p \text{ fixed}
\]

Due to Lemma 4.7 and the approximation properties of \( H_{hp}^1 \times \tilde{H}_{hp}^{1/2} \) the assertion of the theorem follows.

\[
\square
\]

Lemma 4.5 If \( \{(u_{hp}, v_{hp})\}_{hp} \) converges weakly to \( (u, v) \in E = H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \) (for \( h \to 0 \) or \( p \to \infty \)), where \( (u_{hp}, v_{hp}) \in D_{hp} \), then \( (u, v) \in D \).

Proof. Consider \( \phi \in C^0(\bar{\Gamma}_s) \) with \( \phi \geq 0 \). For \( e \in \mathcal{T}_h|_{\Gamma_s} \) we approximate \( \phi \) by a combination of Bernstein polynomials \( B_{e,p} \) on the elements \( e \in \mathcal{T}_h|_{\Gamma_s} \), i.e., with the local mapping \( F_e : \{0, 1\} \to e \): \( t \to x(t) \) we define

\[
\phi_{e,p}(t) := B_{e,p}\phi(F_e(t)) := \sum_{k=0}^{p} \binom{p}{k} t^k (1-t)^{p-k} \phi(F_e(\frac{k}{p})).
\]

With the characteristic function \( \chi_e \) of \( e \in \mathcal{T}_h|_{\Gamma_s} \), we set

\[
\phi_{hp} := B_{\mathcal{T}_h,p}\phi := \sum_{e \in \mathcal{T}_h|_{\Gamma_s}} \chi_e(x) \phi_{e,p}(F_e^{-1}(x)),
\]

yielding \( \phi_{hp} \in \tilde{H}_{hp}^{1/2} \). Due to the construction we have that \( \phi_{hp}\big|_{e \in \mathcal{T}_h|_{\Gamma_s}} \) is a polynomial of degree \( p \) and since the Bernstein operators are monotone we have \( \phi_{hp} \geq 0 \). With [99, Theorem 2.3] we get

\[
\lim_{p \to \infty} \| \phi - \phi_{hp} \|_{L^\infty(\Gamma_s)} = 0, \text{ if } h \text{ fixed.}
\]

(4.32)

If \( p \) is fixed we have by the uniform continuity of \( \phi \in C^0(\bar{\Gamma}_s) \) that

\[
\lim_{h \to 0} \| \phi - \phi_{hp} \|_{L^\infty(\Gamma_s)} = 0, \text{ if } p \text{ fixed.}
\]

(4.33)

Since the embedding \( \tilde{H}_{hp}^{1/2}(\Gamma_s) \subset L^1(\Gamma_s) \) is weakly continuous, \( v_{hp} \in \tilde{H}_{hp}^{1/2} \) converge weakly to \( v \) in \( L^1(\Gamma_s) \) and \( v_{hp} \) is bounded. Therefore, there holds the estimate

\[
| \int_{\Gamma_s} (v_{hp}\phi_{hp-1} - v\phi) \, ds | \leq \| v_{hp} \|_{L^1(\Gamma_s)} \| \phi_{hp-1} - \phi \|_{L^\infty(\Gamma_s)} + \| v \|_{L^1(\Gamma_s)} \int_{\Gamma_s} (v_{hp} - v) \phi \, ds |
\]

and using (4.33), (4.32) and \( \phi \in L^\infty(\Gamma_s) = (L^1(\Gamma_s))^\prime \), we obtain that

\[
\lim_{p \to \infty} \int_{\Gamma_s} v_{hp}\phi_{hp-1} \, ds = \int_{\Gamma_s} v\phi \, ds, \text{ if } h \text{ fixed}
\]
and
\[
\lim_{h \to 0} \int_{\Gamma_s} v_{hp} \phi_{h,p-1} ds = \int_{\Gamma_s} v \phi ds, \text{ if } p \text{ fixed.}
\]
The function \(v_{hp} \phi_{h,p-1}|_{\Gamma_s} \), \(e \in T_h|_{\Gamma_s}\), is a polynomial of degree \(2p - 1\). Therefore, the numerical quadrature using \(p + 1\) Gauss-Lobatto nodes is exact, i.e., with (4.26) we get for all \(e \in T_h|_{\Gamma_s}\)
\[
\int_e v_{hp} \phi_{h,p-1} ds = \frac{|e|}{2} \sum_{j=0}^p [\phi_{h,p-1}(\xi_{j+1})^2 + 1) \rho_j] \geq 0. \quad (4.34)
\]
The inequality follows since \(\phi_{h,p-1}|_{\Gamma_s} \geq 0\) and \(v_{hp}(x) \geq 0\) for all \(x \in G_{hp}\), due to the definition of \(D_{hp}\). Furthermore, it is known that the weights \(\rho_j\) of the Gauss-Lobatto quadrature formula are positive.
Combining both results we obtain that for all \(\phi \in C^0(\Gamma_s)\) with \(\phi \geq 0\)
\[
\int_e v \phi ds \geq 0 \quad \forall e \in T_h|_{\Gamma_s},
\]
hence \(v \geq 0\). We have also \(u_{hp} \in H^1_{hp} \subset H^1(\Omega)\), i.e., \(u \in H^1(\Omega)\). In case of \(n = 2\) we have for all \((u_{hp}, v_{hp})\), that \(\langle S1, u_{hp}|_{\Gamma} + v_{hp} - u_0 \rangle = 0\). Because \(\{(u_{hp}, v_{hp})\}_{hp}\) converges weakly to \((u, v) \in E\) we have \(\langle S1, u|_{\Gamma} + v - u_0 \rangle = 0\) and \((u, v) \in D\).

**Lemma 4.6** There exists a subset \(\chi \subset E = H^1(\Omega) \times \dot{H}^{1/2}(\Gamma_s)\) such that \(\bar{\chi} = D\) and mappings \(r_{hp} : \chi \to D_{hp}\) such that for all \((u, v) \in \chi\), \((r_{hp}(u, v))_{hp}\) converges strongly to \((u, v)\) in \(E\) (as \(p \to \infty\), or \(h \to 0\)).

**Proof.** Consider \(\chi := C^\infty(\Omega) \times C^\infty(\Gamma_s) \cap D\). We define \(v_{hp} := B_{T_h,p} v\) and \(\tilde{u}_{hp} := B_{T_h,p} u\).
In case of \(n \geq 3\) let \(u_{hp} := \tilde{u}_{hp}\). In case of \(n = 2\) we set \(u_{hp} := \tilde{u}_{hp} + \gamma\), \(\gamma \in \mathbb{R}\). Due to \((u, v) \in D\) we have \(\langle S1, u|_{\Gamma} + v - u_0 \rangle = 0\). If we define
\[
\gamma = -\frac{1}{\langle S1, 1 \rangle} \langle S1, \tilde{u}_{hp}|_{\Gamma} + v_{hp} - u_0 \rangle
\]
we have \(\langle S1, u_{hp}|_{\Gamma} + v_{hp} - u_0 \rangle = 0\), i.e., \((u_{hp}, v_{hp}) \in D_{hp}\). Note that there holds \(\langle S1, 1 \rangle \neq 0\) (see [23, Proof of Lemma 3.5]). We have
\[
|\gamma| = \frac{1}{\langle S1, 1 \rangle} |\langle S1, \tilde{u}_{hp}|_{\Gamma} + v_{hp} - u_0 \rangle|
\leq \frac{1}{\langle S1, 1 \rangle} \|S1\|_{H^{-1/2}(\Gamma)} \|u|_{\Gamma} - \tilde{u}_{hp}|_{\Gamma} + v - v_{hp}\|_{H^{1/2}(\Gamma)}
\leq \frac{\|S1\|_{H^{-1/2}(\Gamma)}}{\langle S1, 1 \rangle} \left(\|u - \tilde{u}_{hp}\|_{H^1(\Omega)} + \|v - v_{hp}\|_{H^{1/2}(\Gamma)}\right).
\]
Now, \(r_{hp} : H^2(\Omega) \times (H^1_0(\Gamma_s) \cap C^0(\Gamma_s)) \to D_{hp}\) is defined by
\[
r_{hp}(u, v) := (u_{hp}, v_{hp}). \quad (4.35)
\]
Due to [85, Theorem 4.72] (parallelograms) and [85, Remark 4.74] (triangles) there exists an interpolate \( \bar{u}_{hp} := \Pi_{hp} u \in H^1_{hp} \) such that

\[
\| u - \Pi_{hp} u \|_{H^2(\Omega)} \leq C h^{-1} \| u \|_{H^2(\Omega)}.
\]

As shown in [10, Theorem 4.5] there exists a constant \( C \) independent of \( v \) and \( p \) such that

\[
\| B_{T_h,p} v - v \|_{\dot{H}^{1/2}(\Gamma_s)} \leq C h^{1/2} p^{-1/2} \| v \|_{H^1(\Gamma_s)} \quad \forall v \in H^1_0(\Gamma_s)
\]

and thus for all \( v \in \chi \). Therefore it is obvious that \( r_{hp}(u,v) \in D_{hp} \) for all \( (u,v) \in \chi \) and \( r_{hp}(u,v) \) converges strongly to \( (u,v) \in \chi \) if \( p \) tends to infinity and \( h \) is fixed, or if \( h \) tends to zero and \( p \) is fixed. We have \( \bar{\chi} = D \).

Next, we investigate the difference between the auxiliary problem (4.31) and the original discrete problem.

Lemma 4.7 Let \((\bar{u}_{hp}, \bar{v}_{hp}) \in D_{hp}\) solve

\[
A(\bar{u}_{hp}; \bar{v}_{hp}) \geq \lambda(u_{hp}, \bar{v}_{hp}) \quad \forall (u_{hp}, \bar{v}_{hp}) \in D_{hp},
\]

and let \((\hat{u}_{hp}, \hat{v}_{hp}) \in D_{hp}\) solve

\[
A_{hp}(\hat{u}_{hp}; \hat{v}_{hp}) \geq \lambda_{hp}(u_{hp} - \hat{u}_{hp}, v_{hp} - \hat{v}_{hp}) \quad \forall (u_{hp}, v_{hp}) \in D_{hp}.
\]

Then there holds

\[
c \| (\hat{u}_{hp} - \bar{u}_{hp}, \hat{v}_{hp} - \bar{v}_{hp}) \|_{H^1(\Omega) \times H^{1/2}(\Gamma_s)} \leq \text{dist}_{H^{1/2}(\Gamma)} (V^{-1} (I - K) (\hat{u} + \bar{v}, H^{-1/2}_{hp})
\]

\[
+ \text{dist}_{H^{1/2}(\Gamma)} (V^{-1} (I - K) u_0, H^{-1/2}_{hp})
\]

\[
+ \text{dist}_{H^{1/2}(\Gamma)} ((\hat{u} - \bar{u}_{hp}, \hat{v} - \bar{v}_{hp}) \|_{H^1(\Omega) \times H^{1/2}(\Gamma_s)}
\]

Proof. We can rewrite the forms \( A \) and \( \lambda \) in terms of \( A_{hp} \) and \( \lambda_{hp} \) by

\[
A(\bar{u}_{hp}; \bar{v}_{hp}) = A_{hp}(\bar{u}_{hp}; \bar{v}_{hp}) \quad \lambda(\bar{w}_{hp}, \bar{v}_{hp}) = \lambda_{hp}(\bar{w}_{hp}, \bar{v}_{hp})
\]

for all \((u_{hp}, v_{hp}), (w_{hp}, \eta_{hp}) \in H^1 \times \dot{H}^{1/2}_{hp} \), where we have defined \( S_{hp} := i_{hp}^* W + i_{hp}^*(I - K') h^p \).

Adding these inequalities we obtain

\[
- (A_{hp} (\bar{u}_{hp}, \bar{v}_{hp}) - A_{hp} (\hat{u}_{hp} - \bar{u}_{hp}, \hat{v}_{hp} - \bar{v}_{hp})) \geq \lambda_{hp} (\hat{u}_{hp} - \bar{u}_{hp}, \hat{v}_{hp} - \bar{v}_{hp})
\]

Applying Lemma 4.4 we obtain

\[
c \| (\hat{u}_{hp} - \bar{u}_{hp}, \hat{v}_{hp} - \bar{v}_{hp}) \|_{H^1(\Omega) \times H^{1/2}(\Gamma_s)}^2 \leq \| (S - S_{hp}) (\bar{u}_{hp} + \bar{v}_{hp}) + (S - S_{hp}^*) u_0, (\bar{u}_{hp} - \hat{u}_{hp}) \|_{H^{-1/2}(\Gamma)} \times
\]

\[
\| (\bar{u}_{hp} - \hat{u}_{hp}) \|_{H^{-1/2}(\Gamma)} \times
\]

\[
\| (\hat{v}_{hp} - \bar{v}_{hp}) \|_{H^{1/2}(\Gamma)}.
\]
Therefore, we have

$$
\|(\hat{u}_{hp} - \bar{u}_{hp}, \hat{v}_{hp} - \bar{v}_{hp})\|_{H^1(\Omega) \times H^{1/2}(\Gamma_s)} \leq \|(S - S_{hp})(\bar{u}_{hp}|_{\Gamma} + \bar{v}_{hp})\|_{H^{-1/2}(\Gamma)} + \|(S - S_{hp})u_0\|_{H^{-1/2}(\Gamma)}.
$$

Following [18, Lemma 2.4] we have

$$
\|(S - S_{hp})(\bar{u}_{hp}|_{\Gamma} + \bar{v}_{hp})\|_{H^{-1/2}(\Gamma)} \leq c \cdot \text{dist}_{H^{-1/2}(\Gamma)}(V^{-1}(I - K)(\bar{u}_{hp}|_{\Gamma} + \bar{v}_{hp}), H^{-1/2}_{hp})
$$

$$
\leq c \cdot \text{dist}_{H^{-1/2}(\Gamma)}(V^{-1}(I - K)(\bar{u}|_{\Gamma} + \bar{v}), H^{-1/2}_{hp}) + c \cdot \|\bar{u}|_{\Gamma} + \bar{v} - (\bar{u}_{hp}|_{\Gamma} + \bar{v}_{hp})\|_{H^{1/2}(\Gamma)}
$$

$$
\leq c \cdot \|\bar{u} - \bar{u}_{hp}, \bar{v} - \bar{v}_{hp}\|_{H^{1}(\Omega) \times H^{1/2}(\Gamma_s)}.
$$

Analogously, we have

$$
\|(S - S'_{hp})u_0\|_{H^{-1/2}(\Gamma)} \leq c \cdot \text{dist}_{H^{-1/2}(\Gamma)}(V^{-1}(I - K)u_0, H^{-1/2}_{hp}).
$$

Theorem 4.5

Let the sequence of spaces $H^1_{hp} \times \bar{H}^{1/2}_{hp} \times H^{-1/2}_{hp}$ fulfill the approximation property (4.14). Let $(\hat{u}_{hp}, \hat{v}_{hp}) \in D_{hp}$ be the solutions of

$$
\mathcal{A}_{hp}(\hat{u}_{hp}, \hat{v}_{hp}; u_{hp} - \hat{u}_{hp}, v_{hp} - \hat{v}_{hp}) \geq \lambda_{hp} (u_{hp} - \hat{u}_{hp}, v_{hp} - \hat{v}_{hp}) \quad \forall (u_{hp}, v_{hp}) \in D_{hp}. \quad (4.36)
$$

Then the sequence $\{(\hat{u}_{hp}, \hat{v}_{hp})\}_{hp}$ is uniformly bounded.

Proof. Applying Theorem 4.2 and identifying the spaces $E = F = G = H^1(\Omega) \times H^{1/2}(\Gamma_s)$, we obtain

$$
c \cdot \|\bar{u} - \bar{u}_{hp}, \bar{v} - \bar{v}_{hp}\|_{E}^2
\leq \text{dist}_{H^{-1/2}(\Gamma)}(V^{-1}(I - K)(\bar{u}|_{\Gamma} + \bar{v}), H^{-1/2}_{hp})^2 + \text{dist}_{H^{-1/2}(\Gamma)}(V^{-1}(I - K)u_0, H^{-1/2}_{hp})^2
$$

$$
+ 2\|\mathcal{A}(\bar{u}, \bar{v}; \cdot, \cdot) - \lambda\|_{E_E} \inf_{(u,v) \in E} \|\bar{u} - u, \bar{v} - v\|_{E}
$$

$$
+ \inf_{(u_{hp}, v_{hp}) \in D_{hp}} \left\{ \|\bar{u} - u_{hp}, \bar{v} - v_{hp}\|_{E}^2 + 2\|\mathcal{A}(\bar{u}, \bar{v}; \cdot, \cdot) - \lambda\|_{E_E} \|\bar{u} - u_{hp}, \bar{v} - v_{hp}\|_{E_E} \right\}.
$$

Estimating $\inf_{(u,v) \in E} \|\bar{u} - u, \bar{v} - v\|_{E}$ by $\|\bar{u} - \bar{u}_{hp}, \bar{v} - \bar{v}_{hp}\|_{E}$ and applying the approximation property, there are two constants $C_1, C_2$ such that

$$
\|\bar{u} - u_{hp}, \bar{v} - v_{hp}\|_{E} \leq C_1 \|\bar{u} - \bar{u}_{hp}, \bar{v} - \bar{v}_{hp}\|_{E} + C_2. \quad (4.37)
$$

Therefore, we conclude, that $\{(\hat{u}_{hp}, \hat{v}_{hp})\}_{hp}$ is a bounded sequence.

Relation (4.37) implies that $(\hat{u}_{hp}, \hat{v}_{hp})$ is uniformly bounded. Hence there exists a subsequence $(u_{hp(i)}, v_{hp(i)})$, such that $(u_{hp(i)}, v_{hp(i)})$ converges weakly to $(u^*, v^*)$ in $E = H^1(\Omega) \times H^{1/2}(\Gamma_s)$. Due to Lemma 4.5 we have $(u^*, v^*) \in D$. 

\[\square\]

4.2 A posteriori error estimate for FEM-BEM coupling

Here we investigate an a posteriori error estimator for the FEM-BEM coupling problem described in Section 4.1.

First, we have to describe the non-linear behavior of $\mathcal{A}$ in more detail.

We define the bilinear form $A_w(\cdot; \cdot)$ by

$$
A_w(u; v) := \int_{\Omega} \partial(|\nabla w|) \nabla u \cdot \nabla v \, dx \quad (4.38)
$$
where \( \tilde{\varrho}(x) \in \mathbb{R}^{n \times n} \) is the Jacobian of \( x \rightarrow \varrho(|x|)x \), i.e.,
\[
\tilde{\varrho}(x) = \varrho(|x|)I_{n \times n} + \varrho'(|x|) \frac{x \cdot x^t}{|x|} \quad (x \in \mathbb{R}^n).
\]
(4.39)

From the assumptions on \( \varrho \) and the continuity of the differential operators it follows that there exists a constant \( \nu > 0 \) such that
\[
A_w(u; v) \leq \nu \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall u, v, w \in H^1(\Omega).
\]
(4.40)

**Definition 4.4** We define the norm
\[
\|\psi\|_{W} = (\psi, W\psi)_{\Gamma_s}^{1/2} \quad \forall \psi \in \tilde{H}^{1/2}(\Gamma_s)
\]
which is equivalent to the \( \tilde{H}^{1/2}(\Gamma_s) \) norm (see, e.g., [27]).

Also, we define the norm (related to the energy norm of our problem)
\[
\|(u, \phi)\|_{H} = (\|u\|^2_{H^1(\Omega)} + \|\phi\|^2_W)^{1/2}
\]
which is generated by the bilinear form
\[
a(u, \phi; v, \psi) = \int_{\Omega} (\nabla u \cdot \nabla v + uv) \, dx + (\psi, W\phi)_{\Gamma_s}.
\]
(4.41)

We apply the finite element methods (FEM) and the boundary element method (BEM) to compute approximations to \( u \) and \( \phi \), respectively. For this purpose we consider regular triangulations \( \omega_h \) of \( \Omega \) and partitions \( \gamma_h \) of \( \Gamma_s \). Our test and trial spaces are defined as
\[
T_h := \{ v_h : \Omega \to \mathbb{R} ; v_h \text{ p.w. linear on } \omega_h, v_h \in C^0(\Omega) \}, \quad \sigma_h := \{ \psi_h : \Gamma_s \to \mathbb{R} ; \psi_h \text{ p.w. linear on } \gamma_h, \psi_h \in C^0(\Gamma_s) \}. \quad \quad \text{(4.42)}
\]

Due to computational constraints the partition \( \gamma_h \) is currently given by \( \omega_h|\Gamma_s \).

**Remark 4.4** In the general setting of Section 4.1 we used the notation \( H^1_h \) instead of \( T_h \) and \( \tilde{H}^{1/2} \) instead of \( \sigma_h \). The subspace for the tractions \( H^{-1/2}_h \) is given by the space of piecewise constant functions on the mesh \( \omega_h|\Gamma \).

We denote by \( T_{h/2} \) and \( \sigma_{h/2} \) the meshes which are constructed from \( T_h \) by refining every triangle into 4 triangles, by using a refinement scheme retaining shape-regularity (see Algorithm 4.1, \( E_0 \) consists of all edges in \( T_h \)), and from \( \sigma_h \) by dividing every boundary element into two equal parts.

In order to prove our a posteriori error estimator we need the following auxiliary problem \( (\tilde{P}_{h/2}) \).

**Definition 4.5** Let \( \tilde{S}_{h/2} \) be defined by
\[
\tilde{S}_{h/2} := i_{h/2}^* W \gamma i_{h/2}^* + i_{h/2}^* (I - K') k_h (k_h^* V k_h)^{-1} k_h^* (I - K) \gamma i_{h/2}^*, \quad \quad \text{(4.44)}
\]
let \( \tilde{A}_{h/2} \) be defined by
\[
\tilde{A}_{h/2}(u_{h/2}, v_{h/2})(r_{h/2}, s_{h/2}) := DG(u_{h/2}, v_{h/2}) + (\tilde{S}_{h/2}(u_{h/2}|\Gamma + v_{h/2}), r_{h/2}|\Gamma + s_{h/2})
\]
(4.45)
and let $\tilde{\lambda}_{h/2}$ be defined by
\[
\tilde{\lambda}_{h/2}(u_{h/2}, v_{h/2}) := L(u_{h/2}, u_{h/2}| \Gamma + v_{h/2}) + \langle i_{h/2}^{*} (W + (I - K^{T})k_{h}k_{h}^{*}Vk_{h}^{-1}k_{h}^{*}(I - K))u_{0}, u_{h/2}| \Gamma + v_{h/2} \rangle \quad (4.46)
\]
Then, the problem $(\bar{P}_{h/2})$ reads as follows:
\[
\bar{A}_{h/2}(\bar{u}_{h/2}, \bar{v}_{h/2})(u_{h/2} - \bar{u}_{h/2}, v_{h/2} - \bar{v}_{h/2}) \geq \bar{\lambda}_{h/2}(u_{h/2} - \bar{u}_{h/2}, v_{h/2} - \bar{v}_{h/2}) \quad (4.47)
\]
for all $(u_{h/2}, v_{h/2}) \in D_{h/2}$.

**Remark 4.5** $\tilde{S}_{h/2}$ and $S_{h/2}$ (and analogously $\tilde{\lambda}_{h/2}$ and $\lambda_{h/2}$) differ from each other because $V^{-1}$ was only computed relative to the space $H_{h}^{-1/2}$ instead of $H_{h}^{-1/2}$. But $\tilde{S}_{h/2}$ and $S_{h/2}$, resp. $\tilde{\lambda}_{h/2}$ and $\lambda_{h/2}$, coincide for elements in $H_{h}^{1} \times H_{h}^{1/2}$.

If we define $n_{h} := \dim T_{h}$ and $n_{h/2} := \dim T_{h/2}$ then $n := n_{h/2} - n_{h}$ is the number of new nodes in the refined mesh $\omega_{h/2}$. For each of these new nodes $\nu_{i}$, $1 \leq i \leq n$, we define the piecewise linear basis functions $b_{i}$ by
\[
b_{i}(\nu_{j}) := \delta_{i,j} \quad (1 \leq j \leq n + n_{h}) \quad (4.48)
\]
where $\nu_{i}$, $(n < i \leq n + n_{h})$ are the nodes in $\omega_{h}$. With $T_{i} = \text{span}\{b_{i}\}$ we denote the one-dimensional space which is spanned by $b_{i}$.

On the boundary we proceed as follows: Let $m := \dim \sigma_{h}$ be the number of boundary elements $\mu_{i}$ in $\gamma_{h}$, i.e., $\gamma_{h} = \{\mu_{1}, \ldots, \mu_{m}\}$. For $1 \leq i \leq m$ we consider bisections of the element $\mu_{i}$, i.e., $\mu_{i} = \mu_{i,1} \cup \mu_{i,2}$ and $|\mu_{i,1}| = |\mu_{i,2}|$. With each $\mu_{i}$ we associate the hat function $\beta_{i}$ defined as
\[
\beta_{i}(x) := \begin{cases} 
1 & \text{if } x \in \tilde{\mu}_{i,1} \cap \tilde{\mu}_{i,2} \\
0 & \text{if } x \in \Gamma \setminus \mu_{i} \\
\text{linear} & \text{otherwise}
\end{cases} \quad (4.49)
\]
Hence we obtain the following subspace decompositions
\[
T_{h/2} = T_{h} \oplus L_{h}, \quad L_{h} := T_{1} \oplus T_{2} \oplus \ldots \oplus T_{n}, \quad (4.50)
\]
\[
\sigma_{h/2} = \sigma_{h} \oplus \lambda_{h}, \quad \lambda_{h} := \sigma_{1} \oplus \sigma_{2} \oplus \ldots \oplus \sigma_{m}. \quad (4.51)
\]
Let $P_{h} : T_{h/2} \rightarrow T_{h}, P_{i} : T_{h/2} \rightarrow T_{i}, p_{h} : \sigma_{h/2} \rightarrow \sigma_{h}$ and $p_{i} : \sigma_{h/2} \rightarrow \sigma_{i}$ be the Galerkin projections with respect to the bilinear forms $b(\cdot, \cdot)$ and $\langle W(\cdot, \cdot) \rangle$, where $b(\cdot, \cdot)$ is defined as
\[
b(u, v) := \int_{\Omega} (\nabla u \cdot \nabla v + uv) \, dx. \quad (4.52)
\]
For all $u \in T_{h/2}$ we define $P_{h}$ and $P_{i}$ by
\[
b(P_{h}u, v) = b(u, v) \quad \forall v \in T_{h}, \quad (4.53)
\]
\[
b(P_{i}u, v) = b(u, v) \quad \forall v \in T_{i}, \quad (4.54)
\]
and for all $\phi \in \sigma_{h/2}$ we define
\[
\langle Wp_{h}\phi, \psi \rangle = \langle W\phi, \psi \rangle \quad \forall \psi \in \sigma_{h}, \quad (4.55)
\]
\[
\langle Wp_{i}\phi, \psi \rangle = \langle W\phi, \psi \rangle \quad \forall \psi \in \sigma_{i}. \quad (4.56)
\]
Finally we define the two-level additive Schwarz operators

\[ P := P_h + \sum_{j=1}^n P_j \quad \text{and} \quad p := p_h + \sum_{j=1}^m p_i. \] (4.57)

The following lemmas state that the operators \( P \) and \( p \) have bounded condition numbers.

**Lemma 4.8** [78, Lemma 3.1] There are constants \( C_1, C_2 > 0 \) which depend only on the smallest angle of the triangles in \( \omega_H \) and on the diameter of \( \Omega \) such that:

\[ C_1\|v\|^2_{H^1(\Omega)} \leq \|P_h v\|^2_{H^1(\Omega)} + \sum_{i=1}^n \|P_i v\|^2_{H^1(\Omega)} \leq C_2 \|v\|^2_{H^1(\Omega)} \] (4.58)

for all \( v \in T_{h/2} \).

**Lemma 4.9** There are constants \( c_1, c_2 > 0 \) such that:

\[ c_1\|\phi\|^2_{W} \leq \|p_h \phi\|^2_{W} + \sum_{j=1}^m \|p_i \phi\|^2_{W} \leq c_2\|\phi\|^2_{W} \] (4.59)

for all \( \phi \in \sigma_{h/2} \).

**Proof.** Using the arguments from [79].

---

**A posteriori error estimate**

Let \( \omega_0 \) be an initial triangulation of \( \Omega \) and let \( \gamma_0 \) be an initial partition of \( \Gamma_s \). We consider subsequent refinements \( \{\omega_k\}_{k=0}^\infty \) and \( \{\gamma_k\}_{k=0}^\infty \) where \( \omega_{k+1} \) is obtained from \( \omega_k \) by refining certain triangles in \( \omega_k \) and \( \gamma_{k+1} \) is obtained by bisecting certain elements in \( \gamma_k \). Hence, we obtain a sequence of nested spaces \( T_k \times \sigma_k \in T_{k+1} \times \sigma_{k+1} \) (\( k \in \mathbb{N}_0 \)). Let \((u, v)\) be the solution of the variational inequality (4.9) and let \((u_k, v_k)\) be the solution of the corresponding discrete problem (4.18). As previously, let \( n = n(k) \) be the number of new nodes in \( \omega_{k+1} \) and let \( m = m(k) \) be the number of boundary elements on \( \gamma_k \).

We assume the following saturation condition:

**Assumption 4.1** Let \((u, v), (u_h, v_h), (\tilde{u}_{h/2}, \tilde{v}_{h/2})\) be the solutions of (4.9), (4.18) and (4.47), respectively. There exist a parameter \( h_0 \in I \) and a constant \( 0 \leq \kappa < 1 \) such that for all \( h \leq h_0 \):

\[ \|(u - \tilde{u}_{h/2}, v - \tilde{v}_{h/2})\|_H \leq \kappa\|(u - u_h, v - v_h)\|_H. \]

As an immediate consequence of Assumption 4.1 we have

**Lemma 4.10** With Assumption 4.1 we have for the solutions \((u, v)\) of (4.9), \((u_h, v_h)\) of (4.18) and \((\tilde{u}_{h/2}, \tilde{v}_{h/2})\) of (4.47)

\[ (1 - \kappa)\|(u - u_h, v - v_h)\|_H \leq \|(\tilde{u}_{h/2} - u_h, \tilde{v}_{h/2} - v_h)\|_H \leq (1 + \kappa)\|(u - u_h, v - v_h)\|_H \] (4.60)

**Theorem 4.6** Let Assumption 4.1 hold. The refinement of all triangles defining \( T_k := T_h \) gives us \( T_{h/2} \), the refinement of all elements defining \( \sigma_k := \sigma_h \) gives us \( \sigma_{h/2} \). Let
(u_k, v_k) be the solution of (4.18) on T_k × σ_k. Then there are constants ζ_1, ζ_2 > 0 such that

\[ \zeta_1 \left( \sum_{i=1}^{n} \Theta_{i,k}^2 + \theta_k^2 + \sum_{j=1}^{m} \theta_{j,k}^2 \right)^{1/2} \leq \|(u - u_k, v - v_k)\|_{\mathcal{H}} \]

\[ \leq \zeta_2 \left( \sum_{i=1}^{n} \Theta_{i,k}^2 + \theta_k^2 + \sum_{j=1}^{m} \theta_{j,k}^2 \right)^{1/2} \]

(4.61)

where

\[ \Theta_{i,k} := \frac{|\lambda_{h/2}(b_{i,h/2}, 0) - \bar{A}_{h/2}(u_k, v_k; b_{i,h/2}, 0)|}{\|b_{i,h/2}\|_{H^1(\Omega)}} \]

(4.62)

\[ \theta_k := \|p^{(k)}\varepsilon_{h/2}\|_{W} \]

(4.63)

\[ \theta_{j,k} := \|p_j\varepsilon_{h/2}\|_{W} \]

(4.64)

and \( \varepsilon_{h/2} \in K_{v_k} := \{ \psi \in \sigma_{h/2} \mid \psi \geq -v_k \} \) is the solution of the variational inequality

\[ \langle W\varepsilon_{h/2}, \psi - \varepsilon_{h/2} \rangle \geq \lambda_{h/2}(v - \varepsilon_{h/2}, 0) - \bar{A}_{h/2}(u_k, v_k; 0, \psi - \varepsilon_{h/2}) \quad \forall \psi \in K_{v_k}. \]

(4.65)

\( b_{i,h/2} \) are the hat functions corresponding to the new nodes introduced in \( T_{h/2} \).

Proof. We define the form

\[ Q(w - u_k; v) = \int_{\Omega} [\rho(|\nabla w|)\nabla w - \rho(|\nabla u_k|)\nabla u_k - \tilde{\rho}(\nabla u_k) \cdot \nabla (w - u_k)] \cdot \nabla v \, dx \]

(4.66)

for all \( v, w \in T_{h/2} \). Note that

\[ Q(w - u_k; v) = \bar{A}_{h/2}(w, \chi; v, \psi) - \bar{A}_{h/2}(u_k, \xi; v, \psi) \]

\[ - A_w(w - u_k; v) - \langle \bar{S}_{h/2}(w|\Gamma + \chi - (u_k|\Gamma + \xi)), v|\Gamma + \psi \rangle \]

(4.67)

for all \( v, w \in T_{h/2} \) and \( \forall \chi, \psi, \xi \in \sigma_{h/2} \).

Since the function

\[ R : \begin{cases} \mathbb{R}^n & \to \mathbb{R}^n, \\ x & \to \rho(|x|)x \end{cases} \]

is differentiable and \( \tilde{\rho} \) is the Jacobian of \( R \), we obtain

\[ \delta(k) := \frac{\|R(\nabla \tilde{u}_{h/2}) - R(\nabla u_k) - \tilde{\rho}(\nabla u_k) \cdot (\nabla (\tilde{u}_{h/2} - u_k))\|_{L^2(\Omega) \times L^2(\Omega)}}{\|\nabla (\tilde{u}_{h/2} - u_k)\|_{L^2(\Omega) \times L^2(\Omega)}} \to 0 \]

for \( \|\tilde{u}_{h/2} - u_k\|_{H^1(\Omega)} \to 0 \) (which is obvious for \( k \to \infty \) with Lemma 4.10), and

\[ Q(\tilde{u}_{h/2} - u_k; v) \leq \delta(k)\|\tilde{u}_{h/2} - u_k\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \]

(4.68)

The solution \( (\varepsilon_{h/2}, \varepsilon_{h/2}) \in T_{h/2} \times K_{v_k} \) of the local defect problem is now defined by

\[ a(\varepsilon_{h/2}, \varepsilon_{h/2}; \phi - \varepsilon_{h/2}, \psi - \varepsilon_{h/2}) \]

\[ \geq \lambda_{h/2}(\phi - \varepsilon_{h/2}, \psi - \varepsilon_{h/2}) - \bar{A}_{h/2}(u_k, v_k; \phi - \varepsilon_{h/2}, \psi - \varepsilon_{h/2}) \]

\[ \forall (\phi, \psi) \in T_{h/2} \times K_{v_k} \]

(4.69)
where the bilinear form $a(\cdot, \cdot)$ is given in (4.41). (Note: In the linear case we have $\|e_{h/2}/2, \varepsilon_{h/2}\|_T \sim \|\tilde{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k\|_h$ due to Lemma 2.5. For the nonlinear case we have to show this more explicitly.) Note, that this defect problem splits into two separate problems, because $a(\cdot, \cdot)$ is block-diagonal.

The solution $(\tilde{u}_{h/2}, \tilde{v}_{h/2})$ at level $h/2$ satisfies

$$\mathcal{A}_{h/2}(\tilde{u}_{h/2}, \tilde{v}_{h/2}; \phi - \tilde{u}_{h/2}, \psi - \tilde{v}_{h/2}) \geq \hat{\lambda}_{h/2}(\phi - \tilde{u}_{h/2}, \psi - \tilde{v}_{h/2}) \quad \forall (\phi, \psi) \in T_{h/2} \times \sigma_{h/2}, \psi \geq 0$$

and $(u_k, v_k)$ at level $k$ satisfies

$$\mathcal{A}_k(u_k, v_k; \phi - u_k, \psi - v_k) \geq \lambda_k(\phi - u_k, \psi - v_k) \quad \forall (\phi, \psi) \in T_k \times \sigma_k, \psi \geq 0.$$  \hfill{(4.70)}

Due to Lemma 4.4 we have

$$c \cdot \|\|\tilde{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k\|_T^2 \leq \mathcal{A}_{h/2}(\tilde{u}_{h/2}, \tilde{v}_{h/2}; \phi - \tilde{u}_{h/2}, \psi - \tilde{v}_{h/2}) - \hat{\lambda}_{h/2}(u_k, v_k; \tilde{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k) = -\mathcal{A}_{h/2}(\tilde{u}_{h/2}, \tilde{v}_{h/2}; e_{h/2} + u_k - \tilde{u}_{h/2} + v_k - \tilde{v}_{h/2}) + \hat{\lambda}_{h/2}(e_{h/2} + u_k - \tilde{u}_{h/2} + v_k - \tilde{v}_{h/2})$$

$$-\hat{\lambda}_{h/2}(e_{h/2} + u_k - \tilde{u}_{h/2} + v_k - \tilde{v}_{h/2}) - \mathcal{A}_{h/2}(u_k, v_k; \tilde{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k) + \mathcal{A}_{h/2}(\tilde{u}_{h/2}, \tilde{v}_{h/2}; e_{h/2}) \leq 0.$$  \hfill{(4.71)}

Since $e_{h/2} + v_k \geq 0$ we have due to (4.70) that

$$-\mathcal{A}_{h/2}(\tilde{u}_{h/2}, \tilde{v}_{h/2}; e_{h/2} + u_k - \tilde{u}_{h/2} + v_k - \tilde{v}_{h/2}) + \hat{\lambda}_{h/2}(e_{h/2} + u_k - \tilde{u}_{h/2} + v_k - \tilde{v}_{h/2}) \leq 0.$$  \hfill{(4.72)}

Therefore

$$c \cdot \|\|\tilde{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k\|_T^2 \leq -\hat{\lambda}_{h/2}(e_{h/2} + u_k - \tilde{u}_{h/2} + v_k - \tilde{v}_{h/2}) - \mathcal{A}_{h/2}(u_k, v_k; \tilde{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k)$$

$$+ \mathcal{A}_{h/2}(\tilde{u}_{h/2}, \tilde{v}_{h/2}; e_{h/2}) = \hat{\lambda}_{h/2}(e_{h/2} + u_k - \tilde{u}_{h/2} + v_k - \tilde{v}_{h/2}) - \mathcal{A}_{h/2}(u_k, v_k; \tilde{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k - e_{h/2})$$

$$- \mathcal{A}_{h/2}(u_k, v_k; e_{h/2} + v_k) + \mathcal{A}_{h/2}(\tilde{u}_{h/2}, \tilde{v}_{h/2}; e_{h/2}).$$

Since $\tilde{v}_{h/2} - v_k \geq -v_k$, i.e., $\tilde{v}_{h/2} - v_k \in K_{\varepsilon h/2}$, we have due to (4.69) that

$$\hat{\lambda}_{h/2}(e_{h/2} + u_k - \tilde{u}_{h/2} + v_k - \tilde{v}_{h/2}) \leq a(e_{h/2}, e_{h/2}; \tilde{u}_{h/2} - u_k - e_{h/2}, \tilde{v}_{h/2} - v_k - e_{h/2}).$$

Therefore

$$c \cdot \|\|\tilde{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k\|_T^2 \leq a(e_{h/2}, e_{h/2}; \tilde{u}_{h/2} - u_k - e_{h/2}, \tilde{v}_{h/2} - v_k - e_{h/2})$$

$$- \mathcal{A}_{h/2}(u_k, v_k; e_{h/2} + v_k) + \mathcal{A}_{h/2}(\tilde{u}_{h/2}, \tilde{v}_{h/2}; e_{h/2}) = a(e_{h/2}, e_{h/2}; \tilde{u}_{h/2} - u_k - v_k - e_{h/2}) + \mathcal{A}_{h/2}(\tilde{u}_{h/2} - u_k; e_{h/2})$$

$$+ Q(\tilde{u}_{h/2} - u_k; e_{h/2}),$$
Together with the continuity of $a(\cdot, \cdot)$, the continuity of $A_{u_k}$ (see (4.40)) and the boundedness of $Q(\cdot, \cdot)$ we obtain

\[
c \cdot \|(\hat{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k)\|^2_H + \|(e_{h/2}, \varepsilon_{h/2})\|^2_H \\
\leq a(e_{h/2}, \varepsilon_{h/2}; \hat{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k) \\
+ A_{u_k}(\hat{u}_{h/2} - u_k; e_{h/2}) \\
+ \langle \hat{S}_{h/2}(\hat{u}_{h/2} - u_k) \rangle + \langle \tilde{v}_{h/2} - v_k \rangle, e_{h/2} \rangle + \varepsilon_{h/2} + Q(\hat{u}_{h/2} - u_k; e_{h/2}) \\
\leq (1 + c + \delta(k))\|(e_{h/2}, \varepsilon_{h/2})\|_H \|(\hat{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k)\|_H.
\]

Therefore we have

\[
\|(e_{h/2}, \varepsilon_{h/2})\|_H \sim \|(\hat{u}_{h/2} - u_k, \tilde{v}_{h/2} - v_k)\|_H. \tag{4.72}
\]

Since

\[
\|(e_{h/2}, \varepsilon_{h/2})\|^2_H = a(e_{h/2}, \varepsilon_{h/2}; e_{h/2}, \varepsilon_{h/2}) = b(e_{h/2}, e_{h/2}) + (W \varepsilon_{h/2}, \varepsilon_{h/2})
\]

we can apply Lemma 4.8 and Lemma 4.9 to obtain

\[
C_1\|e_{h/2}\|^2_{H^1(\Omega)} + c_1\|\varepsilon_{h/2}\|^2_W \leq \\
\leq \|(P^{(k)}e_{h/2}, p^{(k)}\varepsilon_{h/2})\|^2_H + \sum_{i=1}^n \|P_i e_{h/2}\|^2_{H^1(\Omega)} + \sum_{j=1}^m \|p_j \varepsilon_{h/2}\|^2_W \\
\leq C_2\|e_{h/2}\|^2_{H^1(\Omega)} + c_2\|\varepsilon_{h/2}\|^2_W. \tag{4.73}
\]

where $P^{(k)} : T_{h/2} \rightarrow T_k$ and $P_i : T_{h/2} \rightarrow \text{span}\{h_{i,h/2}\}$ are the Galerkin projections with respect to the bilinear form $b(\cdot, \cdot)$ and $p^{(k)} : \sigma_{h/2} \rightarrow \sigma_k$ and $p_j : \sigma_{h/2} \rightarrow \text{span}\{\beta_{j,h/2}\}$ are the Galerkin projections with respect to the bilinear form $(W \cdot, \cdot)$.

By definition of $P^{(k)}$ and $p^{(k)}$ we have

\[
\|(P^{(k)}e_{h/2}, p^{(k)}\varepsilon_{h/2})\|^2_H = a(e_{h/2}, \varepsilon_{h/2}; P^{(k)}e_{h/2}, p^{(k)}\varepsilon_{h/2}) \\
= a(e_{h/2}, \varepsilon_{h/2}; P^{(k)}e_{h/2}, 0) + a(e_{h/2}, \varepsilon_{h/2}; 0, p^{(k)}\varepsilon_{h/2}).
\]

Due to (4.69) we have

\[
a(e_{h/2}, \varepsilon_{h/2}; \eta P^{(k)}e_{h/2}, 0) \geq \hat{\lambda}_{h/2}(\eta P^{(k)}e_{h/2}, 0) - \hat{A}_{h/2}(u_k, v_k; \eta P^{(k)}e_{h/2}, 0) \quad \forall \eta \in \mathbb{R},
\]

i.e.,

\[
a(e_{h/2}, \varepsilon_{h/2}; P^{(k)}e_{h/2}, 0) = \hat{\lambda}_{h/2}(P^{(k)}e_{h/2}, 0) - \hat{A}_{h/2}(u_k, v_k; P^{(k)}e_{h/2}, 0).
\]

Due to $P^{(k)}e_{h/2} \in T_k$ and (4.71) we have

\[
\hat{\lambda}_{h/2}(\eta P^{(k)}e_{h/2}, 0) - \hat{A}_{h/2}(u_k, v_k; \eta P^{(k)}e_{h/2}, 0) \leq 0 \quad \forall \eta \in \mathbb{R},
\]

i.e.,

\[
a(e_{h/2}, \varepsilon_{h/2}; P^{(k)}e_{h/2}, 0) = \hat{\lambda}_{h/2}(P^{(k)}e_{h/2}, 0) - \hat{A}_{h/2}(u_k, v_k; P^{(k)}e_{h/2}, 0) = 0,
\]

i.e.,

\[
\|(P^{(k)}e_{h/2}, p^{(k)}\varepsilon_{h/2})\|^2_H = a(e_{h/2}, \varepsilon_{h/2}; 0, p^{(k)}\varepsilon_{h/2}).
\]
By definition of $P_i$ we obtain

$$P_i e_{h/2} = \frac{b(e_{h/2}^i, b_{i,h/2})}{b(b_{1,h/2}^i, b_{i,h/2})} b_{i,h/2} = \frac{a(e_{h/2}^i, v_{h/2}^i; b_{i,h/2}^i, 0)}{\|b_{i,h/2}^i\|_{H^1(\Omega)}^2} b_{i,h/2}.$$  

By (4.69) we have

$$a(e_{h/2}, v_{h/2}; \eta b_{i,h/2}, 0) \geq \tilde{\lambda}_{h/2}(\eta b_{i,h/2}, 0) - \tilde{\mathcal{A}}_{h/2}(u_k, v_k; \eta b_{i,h/2}, 0) \forall \eta \in \mathbb{R},$$

which is not very useful. Therefore we have to solve the auxiliary problem for $\Theta_{i,k}$. Combining this with Lemma 4.8 and Lemma 4.9 we obtain the assertion of the theorem.

Correspondingly we have

$$p_j e_{h/2} = \frac{\langle W e_{h/2}, \beta_{j,h/2} \rangle}{\langle W \beta_{j,h/2}, \beta_{j,h/2} \rangle} \beta_{j,h/2} = \frac{a(e_{h/2}, v_{h/2}; 0, \beta_{j,h/2})}{\|\beta_{j,h/2}\|_W^2} \beta_{j,h/2}.$$  

By (4.69) we have

$$a(e_{h/2}, v_{h/2}; 0, \eta \beta_{j,h/2}) \geq \tilde{\lambda}_{h/2}(0, \eta \beta_{j,h/2}, 0) - \tilde{\mathcal{A}}_{h/2}(u_k, v_k; 0, \eta \beta_{j,h/2})$$

for all $\eta \in \mathbb{R}$ with $e_{h/2} + \eta \beta_{j,h/2} \geq -v_k$. Without further knowledge we can only assume $\eta \in \mathbb{R}_{>0}$. This gives the lower estimate

$$a(e_{h/2}, v_{h/2}; 0, \beta_{j,h/2}) \geq \tilde{\lambda}_{h/2}(0, \beta_{j,h/2}) - \tilde{\mathcal{A}}_{h/2}(u_k, v_k; 0, \beta_{j,h/2})$$

which is not very useful. Therefore we have to solve the auxiliary problem for $e_{h/2}$ to compute the indicators. Combining this with Lemma 4.8 and Lemma 4.9 we obtain the assertion of the theorem.

Since the new basis functions $b_{i,h/2} \in T_{h/2} \setminus T_k$ have local support, according to Figure 4.2, the computation of $\Theta_{i,k}$ is simply a postprocessing step, which can be done at rather low costs. The computation of $\theta_{j,k}$ involves the solution of an auxiliary problem consisting of a variational inequality and a dense matrix. But the computation of the Steklov-Poincaré operator has been removed in the auxiliary problem and the inverse of $(k_h^s V k_h)$ which is still needed can be reused from the solution of the problem before.

**Adaptive Algorithm**

For $k \in \mathbb{N}_0$ let $\omega_k$ be a regular triangulation of $\Omega$ and let $\gamma_k$ be a subdivision of $\Gamma_s$. Let $(u_k, v_k) \in T_k \times \sigma_k$ denote the Galerkin solution. We define $N_k := |\omega_k|$. First, we give the algorithm for a shape-regular refinement scheme.

**Algorithm 4.1** Let $\omega_k$ be a conforming mesh. Let $E_k$ be the set of all edges in $\omega_k$. The edges of every triangle $\delta \in \omega_k$ are denoted by $e_{\delta,1}, e_{\delta,2}, e_{\delta,3}$ and the longest edge of $\delta \in \omega_k$ by $e_\delta$. With $E_0 \subset E_k$ we denote an initial set of edges, which have to be refined. Let $i = 0$.

1. Set $i \leftarrow i + 1$. 
2. $E_i$ is defined by

$$E_{i-1} \subseteq E_i.$$ 

For all $\delta \in \omega_k$ do

If $\{e_{\delta,1}, e_{\delta,2}, e_{\delta,3}\} \cap E_i \neq \emptyset$ then let $e_\delta \in E_i$.

end do

3. if $E_i \neq E_{i-1}$ then goto 1.

Now, for all elements $\delta \in \omega_k$ which have at least one edge in $E_i$, the longest edge is also in $E_i$. The new mesh $\omega_{k+1}$ is defined as follows:

For all $\delta \in \omega_k$ do

If $\{e_{\delta,1}, e_{\delta,2}, e_{\delta,3}\} \cap E_i = \emptyset$ then

$\delta \in \omega_{k+1}$

Else

connect the midpoint of the longest edge $e_\delta$ with the opposite vertex.

For all $e \in \{e_{\delta,1}, e_{\delta,2}, e_{\delta,3}\} \cap E_i$ with $e \neq e_\delta$ connect the midpoint of $e_\delta$ with the midpoint of $e$, cf. Figure 4.1.

This creates 2, 3 or 4 new elements in $\omega_{k+1}$.

End if

End do

Due to [83] the newly constructed mesh $\omega_{k+1}$ is a conforming triangulation with the following characteristics:

1. All edges in $E_0$ have been refined in $\omega_{k+1}$.

2. $\omega_{k+1}$ is nested in $\omega_k$ in such a way that each refined triangle is embedded in one triangle of $\omega_k$ in one of the ways shown in Figure 4.1.

3. The triangulation $\omega_{k+1}$ is non-degenerate; namely, the interior angles of all triangles of $\omega_{k+1}$ are guaranteed to be bounded away from 0.

4. $\omega_{k+1}$ is smooth, in the sense that the transition between large and small triangles is not abrupt.

For each triangle $\delta_i \in \omega_k$, $1 \leq i \leq N_k$, we consider the regular refinement $\omega_{i,k}$ of $\omega_k$ which is obtained by refining $\delta_i$ and its neighbors according to Figure 4.2. Note that $\omega_{i,k}$ contains three new nodes $x_{j_1}, x_{j_2}, x_{j_3}$ on the boundary of $\delta_i$. With each node $x_j$ we associate a new basis function $b_{j,k}$ which is one at $x_j$ and zero at all other nodes in $\omega_{i,k}$.

As before we define the quantities

$$\Theta_{j,k} = \frac{[\lambda_{h/2}(b_{j,h/2},0) - \tilde{A}_{h/2}(u_k,v_k; b_{j,h/2},0)]}{\|b_{j,h/2}\|_{H^1(\Omega)}}$$
Figure 4.1: Refinement of \( \delta \in \omega_k \). The longest edge of \( \delta \) is denoted by \( s \). The new nodes are the midpoints of the edges of \( \delta \).

Figure 4.2: Local refinement of \( \delta \in \omega_k \) and of the neighbor elements.

\( (j \in \{j_1, j_2, j_3\}) \) and

\[ \eta_{i,k} := (\Theta_{j_1,k}^2 + \Theta_{j_2,k}^2 + \Theta_{j_3,k}^2) \]

\( (i = 1, \ldots, n_k) \).

We estimate the global error by

\[ \eta_k = \left( \sum_{i=1}^{N_k} \eta_{i,k}^2 + \sum_{j=1}^{M_k} \theta_{j,k}^2 \right)^{1/2}. \quad (4.75) \]

The adaptive refinement algorithm reads as follows:

**Algorithm 4.2** Let the parameters \( \zeta \in [0,1] \), \( \delta > 0 \) and an initial triangulation \( \omega_0 \) of \( \Omega \) be given. With \( T_0, \sigma_0, \tau_0 \) we denote the initial test and trial spaces, representing \( H^1_h, \dot{H}^{1/2}_h, H^{-1/2}_h \), which are induced by the mesh \( \omega_0 \). For \( k = 0, 1, 2, \ldots \) perform
1. Compute the solution \((u_k, v_k) \in T_k \times \sigma_k\) of the variational inequality (4.18).

2. Solve the auxiliary problem \(\langle W\varepsilon_{h/2}, \psi - \varepsilon_{h/2} \rangle \geq \tilde{\lambda}_{h/2}(v - \varepsilon_{h/2}, 0) - \tilde{A}_{h/2}(u_k, v_k; 0, \psi - \varepsilon_{h/2}) \quad \forall \psi \in K_{v_k}\).

3. Compute the error indicators \(\eta_k, \theta_k, \theta_{j,k}\) according to Theorem 4.6. Compute the global error estimate \(\varepsilon_k\). Stop if \(\varepsilon_k < \delta\).

4. The refinement set \(E_0\) contains the edges of triangle \(\delta_i \in \omega_k\) if the local error indicator \(\eta_{h,k}\) belongs to the largest \([(1 - \zeta) \cdot n_k]\) 100\% biggest of all \(\eta_{h,k}\). The refinement set \(E_0\) also contains the edges in \(\omega_k\) corresponding to elements in \(\gamma_k\) if their local error estimate \(\theta_{j,k}\) belongs to the largest \([(1 - \zeta) \cdot m_k]\) 100\% biggest of all \(\theta_{j,k}\). Apply Algorithm 4.1 to \((\omega_k, E_0)\) to create a new and conforming mesh \(T_{k+1} \times \sigma_{k+1} \supset T_k \times \sigma_k\). Goto 1.

4.3 Solvers for FEM-BEM coupling (primal formulation)

4.3.1 Preconditioners for FEM-BEM coupling with Signorini contact

The discretization of the problem given in Section 4.1 leads in the linear case \((\varrho(\cdot) \equiv 1)\) to the following block matrix:

\[
A_h := \begin{pmatrix}
A & B & 0 \\
B^T & C + S_{ST} & S_{ST} \\
0 & S_{ST} & S_{SS}
\end{pmatrix}
\] (4.76)

\(A, B, C\) are the different parts of the FEM-matrix (\(A\) belonging to the interior nodes, \(C\) belonging to all boundary nodes and \(B\) belonging to the coupling of interior and boundary nodes), and \(S\) with its different subscripts is the Steklov-Poincaré operator acting either on whole \(\Gamma\) or \(\Gamma_s\) or both.

**Theorem 4.7** Let \(B\) be the 2-block preconditioner

\[
B := \begin{pmatrix}
B_{ABC} & 0 \\
0 & B_S
\end{pmatrix}
\]

where \(B_{ABC}\) is a preconditioner acting on \(u_h\) and \(B_S\) the preconditioner acting on \(v_h\), i.e., there exist constants \(\theta_A, \Theta_A > 0\) and \(\theta_S, \Theta_S > 0\) such that

\[
\theta_A \leq \frac{v_h^T \begin{pmatrix} A & B \\ B^T & C + S_{ST} \end{pmatrix} u_h}{u_h^T B_{ABC} u_h} \leq \Theta_A \quad \text{and} \quad \theta_S \leq \frac{v_h^T S_{SS} v_h}{v_h^T B_S v_h} \leq \Theta_S \quad \forall (u_h, v_h) \in T_h \times \sigma_h.
\] (4.77)

Then there holds

\[
\frac{c_0}{c_1} \min(\theta_A, \theta_S) \leq \frac{(u_h, v_h)^T A_h(u_h, v_h)}{(u_h, v_h)^T B(u_h, v_h)} \leq \frac{c_1}{c_0} (\Theta_A + \Theta_S) \quad \forall (u_h, v_h) \in T_h \times \sigma_h.
\] (4.78)
Proof. Lemma 4.1 in [23] shows the positive definiteness of \( A \), i.e.,
\[
c_0 \cdot \|(u, v)\|_{H^1(\Omega) \times H^{1/2}(\Gamma_S)}^2 \leq A((u, v); (u, v)) \quad \forall (u, v) \in H^1(\Omega) \times H^{1/2}(\Gamma_S)
\]
with a constant \( c_0 \) independent of \((u, v)\). Together with Lemma 5.1 in [23] (see also [11]) and the continuity there holds also for the discretized bilinear form
\[
c_0 \cdot \|(u_h, v_h)\|_{H^1}^2 \leq A_h((u_h, v_h); (u_h, v_h)) \leq c_1 \|(u_h, v_h)\|_{H^1}^2 \quad \forall (u_h, v_h) \in T_h \times \sigma_h
\]
with constants independent of the discretization.
Therefore we have
\[
\frac{(u_h, v_h)^T A_h(u_h, v_h)}{(u_h, v_h)^T B(u_h, v_h)} \leq \frac{c_1 \|(u_h, v_h)\|_{\tilde{H}}^2}{(u_h, v_h)^T B(u_h, v_h)} \leq \frac{c_1}{c_0} \left( u_h^T \begin{pmatrix} A & B \\ B^T & C + S_{\Gamma_T} \end{pmatrix} u_h + v_h^T S_{SS} v_h \right) \leq \frac{c_1}{c_0} \left( \Theta_A + \Theta_S \right).
\]

Analogously, we have
\[
\frac{(u_h, v_h)^T A_h(u_h, v_h)}{(u_h, v_h)^T B(u_h, v_h)} \geq \frac{c_0 \|(u_h, v_h)\|_{\tilde{H}}^2}{(u_h, v_h)^T B(u_h, v_h)} \geq \frac{c_0}{c_1} \left( u_h^T \begin{pmatrix} A & B \\ B^T & C + S_{\Gamma_T} \end{pmatrix} u_h + v_h^T S_{SS} v_h \right) \geq \frac{c_0}{c_1} \min(\theta_A, \theta_S).
\]

Lemma 4.11

a) Let \( B_{ABC} = B_{MG,ABC} \) be the symmetric V-cycle multigrid preconditioner belonging to \( \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \) + mass matrix, and let \( B_S = B_{MG,S} \) be the multigrid V-cycle preconditioner belonging to \( S_{SS} \). Then there holds
\[
\kappa(B_{MG} A_h) \leq C.
\]

b) Let \( B_{ABC} = B_{BPX,ABC} \) be the BPX preconditioner belonging to \( \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \), and let \( B_S = B_{BPX,S} \) be the BPX preconditioner belonging to \( S_{SS} \). Then there holds
\[
\kappa(B_{BPX} A_h) \leq C \left( 1 + (\log \frac{1}{h})^2 \right).
\]

Proof. a) Due to Petersdorff, Stephan [97] there exist positive constants \( C_1, C_2 \) such that
\[
C_1 \langle S_{SS} v_h, v_h \rangle \leq \langle S_{SS} B_{MG,S} v_h, v_h \rangle \leq C_2 \langle S_{SS} v_h, v_h \rangle \quad \text{for any } v_h \in \sigma_h
\]
(see also Theorem 3.10).
Due to (4.79) we have that
\[
\begin{pmatrix} A & B \\ B^T & C + S_{\Gamma} \end{pmatrix}
\]
is spectrally equivalent to
\[
\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}
\]
+ mass matrix, i.e., we have
\[
u_h^T \begin{pmatrix} A & B \\ B^T & C + S_{\Gamma} \end{pmatrix} u_h \sim ||u_h||^2_{H^1(\Omega)} = u_h^T \begin{pmatrix} A & B \\ B^T & C + S \end{pmatrix} u_h + u_h^T M u_h \forall u_h \in T_h
\]
with
\[
u_h^T M u_h = \int_\omega u_h \tilde{u}_h, \forall u_h, \tilde{u}_h \in T_h.
\]
Then, let $B_{MG,ABC}$ be one symmetric V-cycle multigrid step for
\[
\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} + M
\]
and we have due to [12] that there exist constants $c_1, c_2$ independent of $h$ such that
\[
c_1 u_h^T B_{MG,ABC} u_h \leq u_h^T \begin{pmatrix} A & B \\ B^T & C + S_{\Gamma} \end{pmatrix} u_h \leq c_2 u_h^T B_{MG,ABC} u_h \forall u_h \in T_h.
\]
b) Due to Tran, Stephan [93] there exist positive constants $C_1, C_2$ such that
\[
C_1 (1 + (\log \frac{1}{h})^2) (S_{SS} v_h, v_h) \leq (S_{SS} B_{BPX,ABC} v_h, v_h) \leq C_2 (S_{SS} v_h, v_h)
\]
for any $v_h \in \sigma_h$ (see also Theorem 3.11). Due to the spectral equivalence (4.80) we can apply [16] and we obtain that there exist constants $c_1, c_2$ independent of $h$ such that
\[
c_1 (1 + (\log \frac{1}{h})^2) u_h^T B_{BPX,ABC} u_h \leq u_h^T \begin{pmatrix} A & B \\ B^T & C + S_{\Gamma} \end{pmatrix} u_h \leq c_2 u_h^T B_{BPX,ABC} u_h \forall u_h \in T_h.
\]
4.4 Numerical examples

In the following we present several numerical examples for the interface-problem with and without Signorini interface conditions. All examples share the same geometry (the L-Shape, see Figure 4.3), and the jumps of \( u \) across the interface, i.e., \( u_0 \) and the jump of \( \frac{\partial u}{\partial n} \) across the interface, i.e., \( t_0 \). We also always have vanishing body forces, i.e., \( f = 0 \). The problems differ according to the size of the Signorini-interface. We have \( u_0 = r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2}) \) and \( t_0 = \frac{2}{3}u_0 \). It is known, that there holds \( \Delta u_0 = 0 \), therefore we obtain

\[
0 = (\Delta u_0, 1)_0 = - (\nabla u_0, \nabla 1)_0 + (1, \frac{\partial}{\partial n} u_0),
\]

i.e., \( (1, t_0) = 0 \). Therefore the requirement of Remark 4.2 is fulfilled and every example has a unique solution.

**Example 4.1 (Numerical results without Signorini-Boundary)** This is an interface problem without Signorini conditions. We use the symmetric FEM-BEM coupling method for the discretization. All computations are done using rectangular mesh elements with linear test- and trial functions for the FEM-part. We have tested the MINRES and CG solvers with preconditioning. Preconditioners have been the multigrid algorithm (V-cycle, one pre- and post-smoothing step using damped Jacobi with damping-factor 0.5), the BPX-algorithm and the hierarchical additive Schwarz method, see [102]. Table 4.1 gives the iteration numbers and Table 4.2 the cpu-times for matrix-computation and solvers. The used stopping criterion was that the last relative change of the solution was smaller than \( 10^{-10} \).

\[
\Delta u_i = 0, \ (i = 1, 2)
\]

\[
u_1 = r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2})
\]

\[
u_2 = 0
\]

\[
[u] = u_1 - u_2 = u_1
\]

\[
[\frac{\partial u}{\partial n}] = \frac{\partial}{\partial n} u_1
\]

<table>
<thead>
<tr>
<th>dof</th>
<th>MINRES</th>
<th>CG</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mg</td>
<td>bpx</td>
</tr>
<tr>
<td>21</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>65</td>
<td>22</td>
<td>16</td>
</tr>
<tr>
<td>225</td>
<td>39</td>
<td>17</td>
</tr>
<tr>
<td>833</td>
<td>74</td>
<td>19</td>
</tr>
<tr>
<td>3201</td>
<td>144</td>
<td>19</td>
</tr>
<tr>
<td>12545</td>
<td>278</td>
<td>19</td>
</tr>
<tr>
<td>49665</td>
<td>545</td>
<td>19</td>
</tr>
</tbody>
</table>

Table 4.1: Iteration numbers for MINRES and CG-schemes with various preconditioners
CHAPTER 4. FEM-BEM COUPLING WITH SIGNORINI CONTACT

Table 4.2: CPU-times (sec) for matrix compilation, MINRES and CG-schemes with various preconditioners

Example 4.2 (Numerical results with Signorini-Boundary) This is an interface problem with Signorini conditions on one large edge of the L-Shape. All computations are done using rectangular mesh elements with linear test- and trial functions for the FEM-part. We have tested the preconditioned Polyak algorithm according to Lemma 4.11. Preconditioners have been the multigrid algorithm (V-cycle, one pre- and post-smoothing step using dampened Jacobi with damping-factor 0.5) and the BPX-algorithm. Table 4.3 gives the iteration numbers and condition numbers and Table 4.4 the cpu-times for matrix-computation and solvers, including the unpreconditioned case. The used stopping criterion was that the last relative change of the solution was smaller than $10^{-10}$.

\[
\Delta u_i = 0, \ (i = 1, 2)
\]
\[
u_0 = r^{2/3}\sin\frac{2}{3}(\varphi - \frac{\pi}{2})
\]
\[
[u] = u_1 - u_2 = u_0|\Gamma_t, \ \left[ \frac{\partial u}{\partial n} \right] = \frac{\partial}{\partial n} u_0|\Gamma_t
\]
\[
[u] = u_1 - u_2 \leq u_0|\Gamma_s
\]
\[
\frac{\partial u_1}{\partial n} = \frac{\partial}{\partial n}(u_2 + u_0)|\Gamma_s \leq 0
\]
\[
\Gamma_s = (-0.25, -0.25)(0.25, -0.25)
\]

Table 4.3: Iteration and Condition numbers for Polyak-scheme with various preconditioners
Table 4.4: CPU- times (sec) for matrix compilation, Polyak-scheme with various preconditioners

<table>
<thead>
<tr>
<th>dof</th>
<th>mat</th>
<th>mg</th>
<th>bpx</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>0.006</td>
<td>0.008</td>
<td>0.003</td>
</tr>
<tr>
<td>72</td>
<td>0.022</td>
<td>0.024</td>
<td>0.010</td>
</tr>
<tr>
<td>240</td>
<td>0.089</td>
<td>0.081</td>
<td>0.037</td>
</tr>
<tr>
<td>864</td>
<td>0.396</td>
<td>0.338</td>
<td>0.146</td>
</tr>
<tr>
<td>3264</td>
<td>1.928</td>
<td>1.437</td>
<td>0.730</td>
</tr>
<tr>
<td>12672</td>
<td>14.079</td>
<td>6.675</td>
<td>3.606</td>
</tr>
<tr>
<td>49920</td>
<td>149.107</td>
<td>30.422</td>
<td>17.446</td>
</tr>
</tbody>
</table>
Example 4.3 (Numerical results with Signorini-Boundary) This is an interface problem with Signorini conditions on both large edges of the L-Shape, as shown in Figure 4.3. All computations are done using rectangular mesh elements with linear test- and trial functions for the FEM-part. We have tested the preconditioned Polyak algorithm according to Lemma 4.11. Preconditioners have been the multigrid algorithm (V-cycle, one pre- and post-smoothing step using dampened Jacobi with damping-factor 0.5) and the BPX-algorithm. Tables 4.5, 4.6 and 4.7 give the extreme eigenvalues and condition numbers for the original system, the system with multigrid-preconditioner and the system with BPX-preconditioner. The condition numbers are also shown in Figure 4.4. We note the linear growth of the condition number of the original system, the logarithmic growth for the system with BPX-preconditioner and that the condition numbers for the system with multigrid preconditioner are bounded. Computing times, iteration numbers and numbers of inner and outer restarts of the Polyak algorithm for the original system, the system with multigrid-preconditioner and the system with BPX-preconditioner are given in Tables 4.8, 4.9 and 4.10. The used stopping criterion was that the last relative change of the solution was smaller than $10^{-10}$.

Figure 4.5 finally shows the solution $u_h$ (color plot and surface plot) and $v_h$ for $\dim T_h = 3288$ and $\dim \sigma_h = 40$. 

\[
\begin{align*}
\Delta u_i &= 0, \quad (i = 1, 2) \\
u_0 &= r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2}) \\
[u] &= u_1 - u_2 = u_0|_{\Gamma_s}, \quad [\frac{\partial u_i}{\partial n}] = \frac{\partial}{\partial n} u_0|_{\Gamma_t} \\
[u] &= u_1 - u_2 \leq u_0|_{\Gamma_s} \\
\frac{\partial u_1}{\partial n} &= \frac{\partial}{\partial n} (u_2 + u_0)|_{\Gamma_s} \leq 0 \\
\Gamma_s &= (-0.25, -0.25)(0.25, -0.25) \cup (-0.25, -0.25)(-0.25, 0.25) \\
\end{align*}
\]
CHAPTER 4. FEM-BEM COUPLING WITH SIGNORINI CONTACT

Table 4.5: Extreme eigenvalues of the unpreconditioned matrix

<table>
<thead>
<tr>
<th>N</th>
<th>$\lambda_{\text{min}}$</th>
<th>$\lambda_{\text{max}}$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 + 8</td>
<td>0.1191171</td>
<td>6.8666176</td>
<td>57.645964</td>
</tr>
<tr>
<td>64 + 16</td>
<td>0.0440775</td>
<td>7.5506673</td>
<td>171.30417</td>
</tr>
<tr>
<td>232 + 24</td>
<td>0.0140609</td>
<td>7.8599693</td>
<td>558.99601</td>
</tr>
<tr>
<td>864 + 32</td>
<td>0.0040314</td>
<td>7.9613671</td>
<td>1974.8471</td>
</tr>
<tr>
<td>3288 + 40</td>
<td>0.0010835</td>
<td>7.989920</td>
<td>7373.9834</td>
</tr>
</tbody>
</table>

Table 4.6: Extreme eigenvalues of the Multigrid preconditioner

<table>
<thead>
<tr>
<th>N</th>
<th>$\lambda_{\text{min}}$</th>
<th>$\lambda_{\text{max}}$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 + 8</td>
<td>0.2638324</td>
<td>51.945662</td>
<td>196.8885</td>
</tr>
<tr>
<td>64 + 16</td>
<td>0.2544984</td>
<td>52.189897</td>
<td>205.06963</td>
</tr>
<tr>
<td>232 + 24</td>
<td>0.2484367</td>
<td>52.274680</td>
<td>210.41444</td>
</tr>
<tr>
<td>864 + 32</td>
<td>0.2442467</td>
<td>52.303797</td>
<td>214.14336</td>
</tr>
<tr>
<td>3288 + 40</td>
<td>0.2412018</td>
<td>52.313155</td>
<td>216.88543</td>
</tr>
</tbody>
</table>

Table 4.7: Extreme eigenvalues of the BPX preconditioner

<table>
<thead>
<tr>
<th>N</th>
<th>$\lambda_{\text{min}}$</th>
<th>$\lambda_{\text{max}}$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 + 8</td>
<td>0.3812592</td>
<td>10.995214</td>
<td>28.839212</td>
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<td>64 + 16</td>
<td>0.4267841</td>
<td>15.119538</td>
<td>35.426666</td>
</tr>
<tr>
<td>232 + 24</td>
<td>0.4458868</td>
<td>17.891084</td>
<td>40.124721</td>
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<tr>
<td>864 + 32</td>
<td>0.4544801</td>
<td>19.893254</td>
<td>43.771457</td>
</tr>
<tr>
<td>3288 + 40</td>
<td>0.4585964</td>
<td>21.465963</td>
<td>46.807965</td>
</tr>
</tbody>
</table>

Table 4.8: Computing times and iteration numbers for the unpreconditioned matrix

<table>
<thead>
<tr>
<th>N</th>
<th>time</th>
<th>iter</th>
<th>inner restarts</th>
<th>outer restarts</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 + 8</td>
<td>0.0100000</td>
<td>17</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>64 + 16</td>
<td>0.0000000</td>
<td>35</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>232 + 24</td>
<td>0.0200000</td>
<td>56</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>864 + 32</td>
<td>0.2400002</td>
<td>98</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3288 + 40</td>
<td>1.8400011</td>
<td>175</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>12752 + 48</td>
<td>16.070000</td>
<td>323</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>50120 + 56</td>
<td>124.63000</td>
<td>600</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Figure 4.4: Condition numbers of primal formulation

Table 4.9: Computing times and iteration numbers using the Multigrid preconditioner

<table>
<thead>
<tr>
<th>N</th>
<th>time</th>
<th>iter</th>
<th>inner restarts</th>
<th>outer restarts</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 + 8</td>
<td>0.0300000</td>
<td>18</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>64 + 16</td>
<td>0.0100000</td>
<td>30</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>232 + 24</td>
<td>0.0800000</td>
<td>33</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>864 + 32</td>
<td>0.3299999</td>
<td>35</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3288 + 40</td>
<td>1.5599995</td>
<td>35</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>12752 + 48</td>
<td>7.4099960</td>
<td>38</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>50120 + 56</td>
<td>30.679993</td>
<td>38</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4.10: Computing times and iteration numbers using the BPX preconditioner

<table>
<thead>
<tr>
<th>N</th>
<th>time</th>
<th>iter</th>
<th>inner restarts</th>
<th>outer restarts</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 + 8</td>
<td>0.0000000</td>
<td>17</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>64 + 16</td>
<td>0.0100000</td>
<td>32</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>232 + 24</td>
<td>0.0200000</td>
<td>39</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>864 + 32</td>
<td>0.1700001</td>
<td>41</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3288 + 40</td>
<td>0.7399998</td>
<td>43</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>12752 + 48</td>
<td>5.9300003</td>
<td>81</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>50120 + 56</td>
<td>25.279968</td>
<td>83</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>
Figure 4.5: Coupling problem, primal formulation
Chapter 5

Mixed FEM-BEM coupling with Signorini contact

This chapter is concerned with the dual formulation of the interface problem consisting of a linear partial differential equation with variable coefficients in some bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$ ($n \geq 2$) and the Laplace equation with some radiation condition in the unbounded exterior domain $\Omega_c := \mathbb{R}^n \setminus \overline{\Omega}$. The two problems are coupled by transmission conditions and Signorini contact conditions on the interface $\Gamma = \partial \Omega$. The exterior part of the interface problem is rewritten using a Neumann to Dirichlet mapping (NtD) given in terms of boundary integral operators. The resulting variational formulation becomes a variational inequality with a linear operator. Then, we treat the corresponding numerical approximation and discuss a discretization of the NtD mapping with an appropriate discretization of the Steklov-Poincaré operator. In particular, assuming some abstract approximation properties and a discrete inf-sup condition, we show unique solvability of the discrete scheme and obtain the corresponding a priori error estimate. Next, we prove that these assumptions are satisfied with Raviart-Thomas elements and piecewise constants in $\Omega$, and hat functions on $\Gamma$. For this choice of subspace we prove the reliability of an a posteriori error estimator. We suggest a solver based on a modified Uzawa algorithm and show convergence. Finally we present some numerical results illustrating our theory.

5.1 Formulation of the mixed FEM-BEM coupling

5.1.1 Introduction

In this section we analyze the linear case of the interface problem (4.1)-(4.5) in Chapter 4. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with Lipschitz boundary $\Gamma$. In order to describe mixed boundary conditions, we let $\Gamma = \Gamma_t \cup \Gamma_s$ where $\Gamma_s$ and $\Gamma_t$ are nonempty, disjoint, and open in $\Gamma$. It is not necessary that $\Gamma_t$ and $\Gamma_s$ have a positive distance. Also, we let $n$ denote the unit normal on $\Gamma$ defined almost everywhere pointing from $\Omega$ into $\Omega_c := \mathbb{R}^n \setminus \overline{\Omega}$.

Then, given $f \in L^2(\Omega)$ and a matrix-valued function $\kappa \in C(\overline{\Omega}) \cap C^1(\Omega)$, we consider the linear partial differential equations

$$\text{div}(\kappa \nabla u) + f = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \Delta u = 0 \quad \text{in} \quad \Omega_c,$$

(5.1)
CHAPTER 5. MIXED FEM-BEM COUPLING WITH SIGNORINI CONTACT

with the radiation condition as $|x| \to \infty$

\[
u(x) = \begin{cases} 
  a + o(1) & \text{if } n = 2 \\
  O(|x|^{2-n}) & \text{if } n \geq 3
\end{cases},
\]

(5.2)

where $a \in \mathbb{R}$ is constant for any $u$ but varying with $u$. We assume here that $\kappa$ induces a strongly elliptic differential operator, that is there exists $\alpha > 0$ such that

\[
\alpha \|\zeta\|^2 \leq (\kappa(x)\zeta) \cdot \zeta \quad \forall x \in \Omega, \quad \forall \zeta \in \mathbb{R}^n.
\]

(5.3)

Writing $u_1 := u$ in $\Omega$ and $u_2 := u$ in $\Omega_c$, the tractions on $\Gamma$ are given by the traces $(\kappa \nabla u_1) \cdot n$ and $-\frac{\partial u_2}{\partial n}$ (note that $n$ points into $\Omega_c$). Next, given $u_0 \in H^{1/2}(\Gamma)$ and $t_0 \in H^{-1/2}(\Gamma)$, we consider transmission conditions

\[
u_1 = u_2 + u_0 \quad \text{and} \quad (\kappa \nabla u_1) \cdot n = \frac{\partial u_2}{\partial n} + t_0 \quad \text{on} \quad \Gamma_t,
\]

(5.4)

and Signorini conditions

\[
u_1 \leq u_2 + u_0, \quad (\kappa \nabla u_1) \cdot n = \frac{\partial u_2}{\partial n} + t_0 \quad \text{and} \quad 0 = (\kappa \nabla u_1) \cdot n (u_2 + u_0 - u_1) \quad \text{on} \quad \Gamma_s
\]

(5.5)

and assume that for $n = 2$ holds

\[
\int_{\Omega} f(x) \, dx + \int_{\Gamma} t_0 \, dx = 0.
\]

(5.6)

In this way, we look for $u_1 \in H^1(\Omega)$ and $u_2 \in H^{1}_{\text{loc}}(\Omega_c)$ satisfying (5.1)–(5.6) in a weak form.

In this section we study this problem in a dual formulation, show uniqueness and existence of the dual problem and analyze its numerical treatment. Thereby, in an abstract setting, the problem (5.1) – (5.6) is written as a saddle point problem given on a convex subset, where the original inequality originating in the Signorini condition is transferred to a Lagrangian multiplier.

In Section 5.1.2 we give the dual formulation corresponding to the primal formulation on which the analysis in [23] was based and show its equivalence to the primal formulation. In Section 5.1.3 we introduce the boundary integral method and rewrite the exterior problem, which leads to a dual FEM-BEM minimization problem. Further analysis leads to a equivalent mixed dual FEM-BEM formulation, which is the saddle point problem mentioned before. We show its uniqueness by proving a continuous inf-sup condition. In Section 5.1.4 we treat the general numerical approximation and discuss the problems resulting from an additional approximation of the inverse Steklov-Poincaré operator, which is essential for its numerical computability. Assuming some abstract approximation properties and a discrete inf-sup condition we proof existence and uniqueness of the discrete saddle point problem and show an a priori estimate. In Section 5.1.5 we choose our discrete subspaces (Raviart-Thomas elements and piecewise constants on the domain and hat functions on the boundary) and prove the discrete inf-sup condition for this choice of subspaces. In Section 5.3 we present a modified Uzawa solver and show convergence for our setting. Finally, in Section 5.4 we present some numerical results underlining our theory.
5.1.2 A dual formulation

In this section we give a dual variational formulation of the problem (5.1)–(5.5) in terms of a convex minimization problem and an associated variational inequality.

For a bounded Lipschitz domain \( \Omega \) with boundary \( \Gamma \) we use the usual Sobolev spaces \([59, 66] H^s(\mathbb{R}^n), H^s(\Omega), H_{\text{div}}(\Omega), \) and for the unbounded domain \( \Omega_c \) the spaces \( H^s_{\text{loc}}(\Omega_c) \) and \( H_{\text{loc}}(\text{div}; \Omega_c) \).

**Definition 5.1** On \( \Omega \) and \( \Omega_c \) we have

\[
H(\text{div}; \Omega) = \{ q \in [L^2(\Omega)]^n : \text{div} \ q \in L^2(\Omega) \},
\]

\[
H^s_{\text{loc}}(\Omega_c) = \{ u : u|_\omega \in H^s(\omega) \text{ for any } \omega = \Omega_c \cap B \text{ with } \Omega \subset B \subset \mathbb{R}^n \},
\]

\[
H_{\text{loc}}(\text{div}; \Omega_c) = \{ q \in [L^2(\Omega_c)]^n : \text{div} \ q|_\omega \in L^2(\omega) \forall \omega = \Omega_c \cap B \text{ with } \Omega \subset B \subset \mathbb{R}^n \}
\]

and for the interface \( \Gamma \) we have

\[
H^s(\Gamma) = \{ u|_\Gamma : u \in H^{s+1/2}(\mathbb{R}^n) \} \quad (s > 0),
\]

\[
H^0(\Gamma) = L^2(\Gamma),
\]

\[
H^s(\Gamma) = (H^{-s}(\Gamma))^* \quad (s < 0),
\]

where \((H^{-s}(\Gamma))^*\) is the dual of \( H^{-s}(\Gamma) \). \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( H^{-1/2}(\Gamma) \) and \( H^{1/2}(\Gamma) \) with respect to the \( L^2(\Gamma) \)-inner product, i.e.,

\[
\langle u, v \rangle = \int_\Gamma u \cdot v \, ds \quad \forall u, v \in L^2(\Gamma).
\]

**Definition 5.2 (Primal formulation)** Let \( f \in L^2(\Omega) \), \( u_0 \in H^{1/2}(\Gamma) \), \( t_0 \in H^{-1/2}(\Gamma) \) be given. Define \( \Phi : H^1(\Omega) \times H^1_{\text{loc}}(\Omega_c) \to \mathbb{R} \cup \{ \infty \} \) by

\[
\Phi(v_1, v_2) := \frac{1}{2} \int_\Omega (\kappa \nabla v_1) \cdot \nabla v_2 \, dx + \frac{1}{2} \int_{\Omega_c} |\nabla v_2|^2 \, dx - \int_\Omega f \cdot v_1 \, dx - \int_\Gamma t_0 \cdot v_2 \, ds. \tag{5.7}
\]

The radiation condition and the essential boundary conditions define the subset of admissible functions

\[
C := \{ (v_1, v_2) \in H^1(\Omega) \times H^1_{\text{loc}}(\Omega_c) : v_1|_{\Gamma_1} = v_2|_{\Gamma_1} + u_0|_{\Gamma_1}, \quad v_1 \leq v_2 + u_0 \text{ on } \Gamma_s,
\]

and (for \( n = 2 \) there exists \( a \in \mathbb{R} \) such that) \( v_2 \) satisfies (5.2) \}. \tag{5.8}

The minimization problem \( (P) \) now reads:

Find \( (u_1, u_2) \in C \) with

\[
\Phi(u_1, u_2) = \min_{(v_1, v_2) \in C} \Phi(v_1, v_2). \tag{5.9}
\]

**Theorem 5.1** [23, Proposition 2.2] The primal problem has a unique solution and is equivalent to problem (5.1)–(5.5).

**Remark 5.1** In [23] the primal problem is analyzed for a non-linear differential equation, but the arguments given there are also valid for the case with variable coefficients.

**Remark 5.2** Note, that without further assumptions, any minimizer \((u_1, u_2)\) in \( C \) satisfies \( u_2 \in L^2 \), where

\[
L_2 = \{ v \in H^1_{\text{loc}}(\Omega_c) : \Delta v = 0 \text{ in } H^{-1}(\Omega_c) \text{ and } v \text{ satisfies } (5.2) \}. \tag{5.10}
\]
More generally, for any \((u_1, u_2) \in C\) there exists \(\tilde{u}_2 \in L_2\) such that \(\tilde{u}_2|_\Gamma = u_2|_\Gamma\), hence \((u_1, \tilde{u}_2) \in C\) and \(\Phi(u_1, \tilde{u}_2) \leq \Phi(u_1, u_2)\). This means that minimization of \(\Phi\) on \(C\) is equivalent to minimization of \(\Phi\) on the non-void set \(\{(u_1, u_2) \in C : u_2 \in L_2\}\).

The space \(L_2\) is, e.g., analyzed in [28] and gives an appropriate frame for the treatment of the exterior problem via boundary integral operators.

Remark 5.3 For \(n = 2\) let \(u = (u_1, u_2) \in C\) be a minimizer of \(\Phi\) in \(C\) and let \(c\) be some constant function. Then we have \(u + c \in C\). Because of \(\Phi(u + c) = \Phi(u) - c \cdot (\int_\Omega f(x) \, dx + \langle t_0, 1 \rangle)\) there has to hold \(\int_\Omega f(x) \, dx + \langle t_0, 1 \rangle = 0\), i.e., \((5.6)\) is satisfied. Otherwise there exists no minimizer.

Lemma 5.1 [23, Lemma 3.4] Let \(u, v \in L_2\), satisfying \((5.2)\), then there holds by Green’s formula
\[
\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Gamma u \cdot \frac{\partial v}{\partial \nu} \, ds. \tag{5.11}
\]

Lemma 5.2 [23, Lemma 3.6] Let \(n = 2\) and let \(u \in L_2\) satisfy \((5.2)\) with constant \(a \in \mathbb{R}\). Then the following assertions are equivalent:
\[\begin{align*}
(i) \quad & \nabla u \in L^2(\Omega_c) \\
(ii) \quad & \langle 1, \frac{\partial u}{\partial n} \rangle = 0.
\end{align*}\]

We shall write \(\mu \leq 0\) on \(\Gamma_s\) for a functional \(\mu \in H^{-1/2}(\Gamma_s^+),\) if
\[\langle \mu, v \rangle \leq 0 \quad \forall v \in \tilde{H}^{1/2}(\Gamma_s^+), \quad v \geq 0.\]

Definition 5.3 (Dual formulation) For all \((p_1, p_2) \in H(\text{div}; \Omega) \times H_{\text{loc}}(\text{div}; \Omega_c)\) we define the convex and coercive functional \(\tilde{\Phi}\) by
\[
\tilde{\Phi}(p_1, p_2) := \frac{1}{2} \int_\Omega (\kappa^{-1} p_1) \cdot p_1 \, dx + \frac{1}{2} \|p_2\|_{L^2(\Omega_c)}^2 - \langle p_1 \cdot n, u_0 \rangle \tag{5.12}
\]
and the subset of admissible functions by
\[
\tilde{C} := \{(p_1, p_2) \in H(\text{div}; \Omega) \times H_{\text{loc}}(\text{div}; \Omega_c) : \quad p_1 \cdot n = p_2 \cdot n + t_0 \text{ on } \Gamma, \quad p_1 \cdot n \leq 0 \text{ on } \Gamma_s, \quad -\text{div} p_1 = f \text{ in } \Omega, \quad \text{div} p_2 = 0 \text{ in } \Omega_c\}. \tag{5.13}
\]

The minimization problem now reads:
Find \((q_1, q_2) \in \tilde{C}\) with
\[
\tilde{\Phi}(q_1, q_2) = \min_{(p_1, p_2) \in \tilde{C}} \tilde{\Phi}(p_1, p_2). \tag{5.14}
\]

Theorem 5.2 The dual problem has a unique solution.

Proof. First, we note that the set \(\tilde{C}\) is convex but it is not closed. On the other hand, we see that for any minimizer \((q_1^0, q_2^0)\) in \(\tilde{C}\) there holds \(q_2^0 = \nabla w\) with some \(w \in L_2\).

Now, since \(L_2\) is convex and closed and \(\tilde{\Phi}\) is convex and coercive on \(\tilde{C}\) the assertion of the theorem follows due to standard arguments, cf. [34].

The following theorem describes the relation between the primal and dual problems.

Theorem 5.3 Let \((u_1, u_2) \in C\) and \((q_1^0, q_2^0) \in \tilde{C}\) be the solutions of the primal and the dual problems, respectively. Then there holds
\[
(q_1^0, q_2^0) = (\kappa \nabla u_1, \nabla u_2) \quad \text{and} \quad \tilde{\Phi}(q_1^0, q_2^0) + \Phi(u_1, u_2) = 0. \tag{5.15}
\]
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Proof. Let us denote 
\[ M := [L^2(\Omega)]^n \times [L^2(\Omega_c)]^n, \]
and let 
\[ J : C \times M \times M \to \mathbb{R} \cup \{\infty\} \]
be the functional defined by
\[
J((v_1, v_2), (N_1, N_2), (q_1, q_2)) := \frac{1}{2} \int_{\Omega} (\kappa N_1) \cdot N_1 \, dx - (f, v_1)_{L^2(\Omega)} + (q_1, \nabla v_1 - N_1)_{[L^2(\Omega)]^n} \\
+ \frac{1}{2} \|N_2\|^2_{[L^2(\Omega_c)]^n} - \langle t_0, v_2 \rangle + (q_2, \nabla v_2 - N_2)_{[L^2(\Omega_c)]^n}
\]
for all \((v_1, v_2) \in C, (N_1, N_2) \in M, (q_1, q_2) \in M\).

We easily find that
\[
\sup_{q_1 \in [L^2(\Omega)]^n} (q_1, \nabla v_1 - N_1)_{[L^2(\Omega)]^n} = \begin{cases} 0 & \text{for } N_1 = \nabla v_1, \\ +\infty & \text{for } N_1 \neq \nabla v_1, \end{cases}
\]
and
\[
\sup_{q_2 \in [L^2(\Omega_c)]^n} (q_2, \nabla v_2 - N_2)_{[L^2(\Omega_c)]^n} = \begin{cases} 0 & \text{for } N_2 = \nabla v_2, \\ +\infty & \text{for } N_2 \neq \nabla v_2, \end{cases}
\]
whence (5.7) gives
\[
\inf_{(v_1, v_2) \in C} \Phi(v_1, v_2) = \inf_{[(v_1, v_2), (N_1, N_2)] \in C \times M} \sup_{(q_1, q_2) \in M} J((v_1, v_2), (N_1, N_2), (q_1, q_2)). \tag{5.16}
\]

We now denote
\[
S_0(q_1, q_2) := \inf_{[(v_1, v_2), (N_1, N_2)] \in C \times M} J((v_1, v_2), (N_1, N_2), (q_1, q_2)) \tag{5.17}
\]
and observe that
\[
S_0(q_1, q_2) \leq \inf_{(v_1, v_2) \in C} J((v_1, v_2), (\nabla v_1, \nabla v_2), (q_1, q_2)) \\
= \inf_{(v_1, v_2) \in C} \Phi(v_1, v_2) = \Phi(u_1, u_2) \quad \forall (q_1, q_2) \in M, \tag{5.18}
\]
which yields
\[
\sup_{(q_1, q_2) \in M} S_0(q_1, q_2) \leq \Phi(u_1, u_2). \tag{5.19}
\]

Furthermore, we can split
\[
S_0(q_1, q_2) = \inf_{(N_1, N_2) \in M} J_1((N_1, N_2), (q_1, q_2)) + \inf_{(v_1, v_2) \in C} J_2((v_1, v_2), (q_1, q_2)) \tag{5.20}
\]
with
\[
J_1((N_1, N_2), (q_1, q_2)) = \frac{1}{2} \int_{\Omega} (\kappa N_1) \cdot N_1 \, dx - (q_1, N_1)_{[L^2(\Omega)]^n} \tag{5.21}
\]
and
\[
J_2((v_1, v_2), (q_1, q_2)) = (q_1, \nabla v_1)_{[L^2(\Omega)]^n} - (f, v_1)_{L^2(\Omega)} + (q_2, \nabla v_2)_{[L^2(\Omega_c)]^n} - \langle t_0, v_2 \rangle.
\]

Next, it is easy to see that
\[
\inf_{(N_1, N_2) \in M} J_1((N_1, N_2), (q_1, q_2)) = -\frac{1}{2} \int_{\Omega} (\kappa^{-1} q_1) : q_1 \, dx - \frac{1}{2} \|q_2\|^2_{[L^2(\Omega_c)]^n}, \tag{5.21}
\]
and then, using Lemma 5.3, we can write
\[
S_0(q_1, q_2) = \left\{ -\frac{1}{2} \int_{\Omega} (\kappa^{-1} q_1) \cdot q_1 \, dx - \frac{1}{2} \|q_2\|_{L^2(\Omega_c)}^2 + \langle q_1 \cdot n, u_0 \rangle = -\hat{\Phi}(q_1, q_2) \quad \text{for} \quad (q_1, q_2) \in \hat{C} \right\} 
\]
\[\text{for} \quad (q_1, q_2) \notin \hat{C}.
\]
Therefore we have
\[
\Phi(u_1, u_2) \geq \sup_{(q_1, q_2) \in M} S_0(q_1, q_2) = \sup_{(q_1, q_2) \in \hat{C}} [-\hat{\Phi}(q_1, q_2)]
\]
\[= - \inf_{(q_1, q_2) \in \hat{C}} \hat{\Phi}(q_1, q_2) = -\hat{\Phi}(q_1^0, q_2^0).
\]
We show now that the functional \(-\hat{\Phi}\) assumes its maximum at \((\hat{q}_1, \hat{q}_2) := (\kappa \nabla u_1, \nabla u_2)\), which, according to the uniqueness of solution for the dual problem, will imply that \((\kappa \nabla u_1, \nabla u_2) = (q_1^0, q_2^0)\). Indeed, since the primal problem is equivalent to the original one, there holds
\[
\text{div}(\kappa \nabla u_1) = -f \quad \text{in} \quad \Omega, \quad \Delta u_2 = 0 \quad \text{in} \quad \Omega_c,
\]
\[
(\kappa \nabla u_1) \cdot n = \frac{\partial u_2}{\partial n} + t_0 \quad \text{on} \quad \Gamma_s, \quad \text{and} \quad (\kappa \nabla u_1) \cdot n \leq 0 \quad \text{on} \quad \Gamma_s,
\]
which is, respectively,
\[
\text{div} \hat{q}_1 = -f \quad \text{in} \quad \Omega, \quad \text{div} \hat{q}_2 = 0 \quad \text{in} \quad \Omega_c,
\]
\[
\hat{q}_1 \cdot n = \hat{q}_2 \cdot n + t_0 \quad \text{on} \quad \Gamma_s, \quad \text{and} \quad \hat{q}_1 \cdot n \leq 0 \quad \text{on} \quad \Gamma_s,
\]
and hence \((\hat{q}_1, \hat{q}_2) \in \hat{C}\).
Now, using that \(0 = (\kappa \nabla u_1) \cdot n (u_2 + u_0 - u_1)\) on \(\Gamma_s\) and that \(u_1 = u_2 + u_0\) on \(\Gamma_1\), we get
\[
\langle (\kappa \nabla u_1) \cdot n, u_1 - u_2 - u_0 \rangle = 0, \tag{5.23}
\]
and since
\[
\langle (\kappa \nabla u_1) \cdot n, u_1 \rangle = (\kappa \nabla u_1, \nabla u_1)_{L^2(\Omega)} + (\text{div}(\kappa \nabla u_1), u_1)_{L^2(\Omega)}
\]
\[= \int_{\Omega} (\kappa \nabla u_1) \nabla u_1 \, dx - (f, u_1)_{L^2(\Omega)} \tag{5.24}
\]
and (for \(u_2 \in \mathcal{L}_2\))
\[
\langle \frac{\partial u_2}{\partial n}, u_2 \rangle = -|u_2|_{H^1(\Omega_c)}^2, \tag{5.25}
\]
we obtain
\[
(f, u_1)_{L^2(\Omega)} = \int_{\Omega} (\kappa \nabla u_1) \nabla u_1 \, dx - \langle (\kappa \nabla u_1) \cdot n, u_2 + u_0 \rangle
\]
\[= \int_{\Omega} (\kappa \nabla u_1) \nabla u_1 \, dx - \left( \frac{\partial u_2}{\partial n} + t_0, u_2 \right) - \langle (\kappa \nabla u_1) \cdot n, u_0 \rangle
\]
\[= \int_{\Omega} (\kappa \nabla u_1) \nabla u_1 \, dx + |u_2|_{H^1(\Omega_c)}^2 - \langle t_0, u_2 \rangle - \langle (\kappa \nabla u_1) \cdot n, u_0 \rangle.
\]
It follows that
\[
-(f, u_1)_{L^2(\Omega)} - \langle t_0, u_2 \rangle = - \int_{\Omega} (\kappa \nabla u_1) \nabla u_1 \, dx - |u_2|_{H^1(\Omega_c)}^2 + \langle \hat{q}_1 \cdot n, u_0 \rangle, \tag{5.26}
\]
and therefore
\[
\Phi(u_1, u_2) = \frac{1}{2} \int_{\Omega} (\kappa \nabla u_1) \nabla u_1 \, dx + \frac{1}{2} |v_2|^2_{H^1(\Omega_c)} - (f, u_1)_{L^2(\Omega)} - \langle t_0, u_2 \rangle = -\bar{\Phi}(\hat{q}_1, \hat{q}_2),
\]
which ends the proof.

\[\square\]

Lemma 5.3 Let \( J_2 : (H^1(\Omega) \times H^1_{\text{loc}}(\Omega_c)) \times ([L^2(\Omega)]^n \times [L^2(\Omega_c)]^n) \to \mathbb{R} \cup \{\infty\} \) be the functional defined by
\[
J_2((v_1, v_2), (q_1, q_2)) := (q_1, \nabla v_1)_{[L^2(\Omega)]^n} - (f, v_1)_{L^2(\Omega)} + (q_2, \nabla v_2)_{[L^2(\Omega_c)]^n} - \langle t_0, v_2 \rangle \quad \text{for all } (v_1, v_2) \in H^1(\Omega) \times H^1_{\text{loc}}(\Omega_c), \quad (q_1, q_2) \in [L^2(\Omega)]^n \times [L^2(\Omega_c)]^n.
\]
Then we have
\[
\inf_{(v_1, v_2) \in C} J_2((v_1, v_2), (q_1, q_2)) = \begin{cases} 
\langle q_1 \cdot n, u_0 \rangle & \text{for } (q_1, q_2) \in \hat{C}, \\
-\infty & \text{for } (q_1, q_2) \notin \hat{C}.
\end{cases}
\]

Proof. Since \( u_0 \in H^{1/2}(\Gamma) \), the extension theorem yields the existence of \( U_0 \in H^1(\Omega) \) such that \( U_0 = u_0 \) on \( \Gamma \). Then we define
\[
C_0 := \{ (u_1, u_2) \in H^1(\Omega) \times H^1_{\text{loc}}(\Omega_c) : u_1 = u_2 \quad \text{on} \quad \Gamma, \quad u_1 \leq u_2 \quad \text{on} \quad \Gamma_s, \\
\text{and for } n = 2 \text{ there exist } a \in \mathbb{R} \text{ such that } u_2 \text{ satisfies } (5.2) \},
\]
and observe that \((v_1, v_2) \in C\) if and only if there exists \((w_1, w_2) \in C_0\) such that \((v_1, v_2) = (w_1 + U_0, w_2)\). In addition, given \((q_1, q_2) \in [L^2(\Omega)]^n \times [L^2(\Omega_c)]^n\), \( J_2((\cdot, \cdot), (q_1, q_2)) \) is a continuous linear functional in \( H^1(\Omega) \times H^1_{\text{loc}}(\Omega_c) \), and hence
\[
\inf_{(v_1, v_2) \in C} J_2((v_1, v_2), (q_1, q_2)) = J_2((U_0, 0), (q_1, q_2)) + \inf_{(w_1, w_2) \in C_0} J_2((w_1, w_2), (q_1, q_2)).
\]
Now, it is not difficult to see that \((q_1, q_2) \in \hat{C}\) if and only if \( J_2((w_1, w_2), (q_1, q_2)) \geq 0 \) for all \((w_1, w_2) \in C_0\). In fact, the first implication follows straightforward from the definition of \( \hat{C} \). Conversely, let \((q_1, q_2) \in [L^2(\Omega)]^n \times [L^2(\Omega_c)]^n\) such that \( J_2((w_1, w_2), (q_1, q_2)) \geq 0 \) \( \forall (w_1, w_2) \in C_0 \), that is
\[
(q_1, \nabla w_1)_{[L^2(\Omega)]^n} - (f, w_1)_{L^2(\Omega)} + (q_2, \nabla w_2)_{[L^2(\Omega_c)]^n} - \langle t_0, w_2 \rangle \geq 0 \quad \forall (w_1, w_2) \in C_0.
\]
Substituting \( w_1 = \pm \varphi \in C^0_\infty(\Omega) \) and \( w_2 = \pm \varphi \in C^0_\infty(\Omega_c) \) in the above inequality we obtain
\[
\text{div } q_1 = -f \text{ in } \Omega \text{ and div } q_2 = 0 \text{ in } \Omega_c,
\]
and hence \((q_1, q_2) \in H(\text{div}; \Omega) \times H_{\text{loc}}(\text{div}; \Omega_c)\). Further, we have
\[
J_2((w_1, w_2), (q_1, q_2)) = (q_1, \nabla w_1)_{[L^2(\Omega)]^n} - (f, w_1)_{L^2(\Omega)} + (q_2, \nabla w_2)_{[L^2(\Omega_c)]^n} - \langle t_0, w_2 \rangle
\]
\[
= (q_1, \nabla w_1)_{[L^2(\Omega)]^n} + (\text{div } q_1, w_1)_{L^2(\Omega)} + (q_2, \nabla w_2)_{[L^2(\Omega_c)]^n} - \langle t_0, w_2 \rangle
\]
\[
= \langle q_1 \cdot n, w_1 \rangle - \langle q_2 \cdot n + t_0, w_2 \rangle,
\]
and then for all \((w_1, w_2) \in C_0\) there holds
\[
0 \leq J_2((w_1, w_2), (q_1, q_2)) = \langle q_1 \cdot n, w_1 \rangle - \langle q_2 \cdot n + t_0, w_2 \rangle.
\]
It follows that \( q_1 \cdot n = q_2 \cdot n + t_0 \) on \( \Gamma \) and \( q_1 \cdot n \leq 0 \) on \( \Gamma_s \), and therefore \((q_1, q_2) \in \hat{C}\). Next, given \((q_1, q_2) \in \hat{C}\) we have
\[
J_2((v_1, v_2), (q_1, q_2)) = \langle q_1 \cdot n, v_1 \rangle - \langle q_2 \cdot n + t_0, v_2 \rangle = \langle q_1 \cdot n, v_1 - v_2 \rangle,
\]
and, since \( v_1 - v_2 = u_0 \) on \( \Gamma_t \), we get
\[
\int_{\Gamma_t} q_1 \cdot n (v_1 - v_2) \, ds = \int_{\Gamma_t} (q_1 \cdot n) u_0 \, ds.
\]
Thus, using that \( q_1 \cdot n \leq 0 \) on \( \Gamma_s \) and \( (v_1 - v_2) \leq u_0 \) on \( \Gamma_s \), we deduce that the infimum will be obtained for the upper bound of \( v_1 - v_2 \), that is
\[
\inf_{(v_1,v_2) \in C} J_2((v_1,v_2),(q_1,q_2)) = (q_1 \cdot n, u_0).
\]
On the other hand, given \( (q_1,q_2) \notin \tilde{C} \), the above characterization result implies that there exists \( (w_1^0, w_2^0) \in C_0 \) such that \( J_2((w_1^0, w_2^0), (q_1,q_2)) < 0 \). Then for all \( m > 0 \) we have \( m (w_1^0, w_2^0) \in C_0 \) and \( \lim_{m \to +\infty} J_2(m(w_1^0, w_2^0), (q_1,q_2)) = -\infty \), whence the corresponding infimum is \(-\infty\).

5.1.3 Boundary integral operators and the exterior problem

In this section we briefly recall some properties of boundary integral operators of the Laplacian, which have been introduced in Definition 3.2, see also Lemma 3.1, and we rewrite the exterior problem.

If \( \text{diam}(\Omega) \) is small enough (for \( n = 2 \)), \( V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \) is strongly positive definite, and therefore it is continuously invertible, i.e., \( V^{-1} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \) is linear, continuous, symmetric, and positive definite and we may define the Steklov-Poincaré operator
\[
S := \frac{1}{2} (W + (K' - I) V^{-1}(K - I)) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).
\]

Lemma 5.4 [23, Lemma 3.5] The Steklov-Poincaré operator \( S \) is linear, symmetric, and positive definite.

Analogously \( W : H^{1/2}(\Gamma)/\mathbb{R} \to H^{-1/2}(\Gamma) \) is strongly positive definite and therefore continuously invertible.

Definition 5.4 The inverse Steklov-Poincaré operator is given by
\[
R := S^{-1} = \frac{1}{2} (V + (I + K) W^{-1}(I + K')) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma).
\]

Remark 5.4 To fix the constant in \( H^{1/2}(\Gamma)/\mathbb{R} \) one usually chooses the subspace of functions with integral mean zero. Unfortunately, this is not optimal from the implementational point of view. Therefore, we add a least squares term to the hypersingular integral operator \( W \). We define
\[
P : \left\{ \begin{array}{c}
H^{1/2}(\Gamma) \to \mathbb{R} \\
\phi \mapsto P\phi = \int_{\Gamma} \phi \, ds
\end{array} \right.
\]
and the positive definite operator
\[
\tilde{W} := W + P' P : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).
\]
Then we can compute \( u := Rt \) for \( t \in H^{-1/2}(\Gamma) \) by computing \( u = \frac{1}{2} (V t + (I + K) \phi) \), where \( \phi \) is the solution of
\[
\tilde{W} \phi = (I + K') t.
\]
Representing the solution $\phi$ of (5.34) like $\phi = \phi_0 + c_\phi$, such that $P\phi_0 = 0$ and $c_\phi \in \mathbb{R}$, we obtain from $(1, \hat{W}\phi) = (1, (I + K't)t)$ and $W1 = 0$ and $K1 = -1$, that $(P1, P\phi) = 0$, and consequently, $c_\phi = 0$ and $\phi = \phi_0 \in H^{1/2}(\Gamma)/\mathbb{R}$ solves also $W\phi = (I + K't)t$ and is unique.

Therefore we can replace $W$ and $H^{1/2}(\Gamma)/\mathbb{R}$ by $\hat{W}$ and $H^{1/2}(\Gamma)$ for the discretization without mentioning it explicitly.

**Lemma 5.5** [28] For $u_2 \in \mathcal{L}_2$ with Cauchy data $(v, \psi)$ and for $z \in \Omega_c$, there holds

$$u_2(z) = \int_\Gamma v(x) \cdot \frac{\partial}{\partial n_x} G(z, x) \, ds_x - \int_\Gamma \psi(x) \cdot G(z, x) \, ds_x + a,$$

where $a$ is the constant appearing in (5.2) for $n = 2$ (and $a = 0$ in (5.35) if $n \geq 3$).

**Lemma 5.6** [23, Lemma 3.3] Let $(v, \psi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. Then the following statements are equivalent.

1. $(v, \psi)$ are Cauchy data for some $u_2 \in \mathcal{L}_2$.
2. $\psi = -S(v - a)$, where $a \in \mathbb{R}$ is the constant in (5.2) for $n = 2$ (and $a = 0$ for $n \geq 3$).

**Definition 5.5 (Dual formulation with boundary integral operators)** Define $\tilde{\Psi} : H(\text{div}; \Omega) \to \mathbb{R} \cup \{\infty\}$ by

$$\tilde{\Psi}(q) := \frac{1}{2} \int_\Omega (\kappa^{-1} q) \cdot q \, dx + \frac{1}{2} \langle q \cdot n, R(q \cdot n) \rangle - \langle q \cdot n, Rt_0 + u_0 \rangle$$

and the subset of admissible functions by

$$\tilde{D} := \{q \in H(\text{div}; \Omega) : q \cdot n \leq 0 \text{ on } \Gamma_s, -\text{div} q = f \text{ in } \Omega\}$$

Then, the problem $(\tilde{P})$ consists in finding a minimizer $q$ in $\tilde{D}$ of $\tilde{\Psi}$:

Find $q \in \tilde{D}$ with

$$\tilde{\Psi}(q) = \min_{v \in \tilde{D}} \tilde{\Psi}(v).$$

**Remark 5.5** For $n = 2$ and for any function $q \in \tilde{D}$ there holds

$$\langle q \cdot n, 1 \rangle = (\text{div} q, 1)_{L^2(\Omega)} + (q, \nabla 1)_{L^2(\Omega)} = -(f, 1)_{L^2(\Omega)} = \langle t_0, 1 \rangle$$

due to Remark 5.3, i.e., $\langle q \cdot n - t_0, 1 \rangle = 0$.

**Theorem 5.4** a) The dual problems $(P)$ and $(\tilde{P})$ are equivalent in the following sense:

1. If $q^D \in \tilde{D}$ is a solution of $(\tilde{P})$, and $q^D_1$ is the gradient of the function $u_2$ given by the representation formula (5.35) with $(R(q^D \cdot n - t_0), q^D \cdot n - t_0)$ replacing $(v, \psi)$, then $(q^D, q^D_1)$ minimizes the functional $\tilde{\Psi}$.

2. If $(q_0^D, q_0^D_1) \in \tilde{C}$ is the minimizer of $\tilde{\Psi}$ on $\tilde{C}$, then $q_0^D$ is a solution of $(\tilde{P})$.

b) $(\tilde{P})$ has a unique solution.
Find or, equivalently (see [34]), \( \bar{v} \) a solution of the following variational inequality:

\[
H \quad \text{and the subset of admissible functions by}
\]

Then, the problem \((M)\) consists in finding a saddle point \((q, u, \lambda)\) of \(H\) in \(H(\text{div}; \Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)\), i.e.,

Find \((\bar{q}, \bar{u}, \bar{\lambda}) \in H(\text{div}; \Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)\) with

or, equivalently (see [34]), find a solution of the following variational inequality:

Find \((\bar{q}, \bar{u}, \bar{\lambda}) \in H(\text{div}; \Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)\) such that

\[
a(\bar{q}, q) + b(q, \bar{u}) + d(q, \bar{\lambda}) = \langle q \cdot n, r \rangle \quad \forall q \in H(\text{div}; \Omega),
\]

\[
b(q, u) = -\int_{\Omega} fu \; dx \quad \forall u \in L^2(\Omega),
\]

\[
d(\bar{q}, \bar{\lambda} - \bar{\lambda}) \leq 0 \quad \forall \bar{\lambda} \in \tilde{H}^{1/2}(\Gamma_s),
\]
where
\[
a(p, q) := \int_{\Omega} (\kappa^{-1} p) \cdot q \, dx + \langle q \cdot n, R(p \cdot n) \rangle \quad \forall p, q \in H(\text{div}; \Omega),
\]
(5.47)
\[
b(q, u) := \int_{\Omega} u \text{div} \, q \, dx \quad \forall (q, u) \in H(\text{div}; \Omega) \times L^2(\Omega),
\]
(5.48)
\[
d(q, \lambda) := \langle q \cdot n, \lambda \rangle_{\Gamma_s} \quad \forall (q, \lambda) \in H(\text{div}; \Omega) \times \tilde{H}^{1/2}(\Gamma_s),
\]
(5.49)
and \( r := R(t_0) + u_0 \).

Next, we introduce the bilinear form
\[
B(q, (u, \lambda)) = b(q, u) + d(q, \lambda) \quad \forall (q, u, \lambda) \in H(\text{div}; \Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s),
\]
(5.50)
and rewrite the variational inequality (5.44)–(5.46) as
\[
a(\hat{q}, q) + B(q, (\hat{u}, \hat{\lambda})) = \langle q \cdot n, r \rangle \quad \forall q \in H(\text{div}; \Omega)
\]
(5.51)
\[
B(\hat{q}, (u - \hat{u}, \lambda - \hat{\lambda})) \leq -\int_{\Omega} f(u - \hat{u}) \, dx \quad \forall (u, \lambda) \in L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s).
\]
(5.52)

Definition 5.7 Let \( \|q\|_a \) be given by
\[
\|q\|_a^2 := a(q, q) \quad \forall q \in H(\text{div}; \Omega).
\]
(5.53)

Remark 5.6 The norm \( \|q\|_a^2 \) is equivalent to \( \|q\|^2_{L^2(\Omega)} + \|q \cdot n\|^2_{H^{-1/2}(\Gamma)} \).

Problems \((\bar{P})\) and \((M)\) are connected as follows.

Theorem 5.5 \((\bar{P})\) and \((M)\) are equivalent in the following sense:

(i) If \((\hat{q}, \hat{u}, \hat{\lambda}) \in H(\text{div}; \Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)\) is a saddle point of \( \mathcal{H} \) in \( H(\text{div}; \Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \), then \( \hat{q} = \kappa \nabla \hat{u}, \hat{u} = R(t_0 - \hat{q} \cdot n) + u_0 \) on \( \Gamma_t \), \( \hat{\lambda} = -R(\hat{q} \cdot n - t_0) + u_0 - \hat{u} \) on \( \Gamma_s \), and \( \hat{q} \in \bar{D} \) is the solution of problem \((\bar{P})\).

(ii) Let \( q^D \in \bar{D} \) be the solution of \((\bar{P})\), and define \( \lambda := -R(q^D \cdot n - t_0) + u_0 - \hat{u} \) on \( \Gamma \), where \( \hat{u} \in H^{1}(\Omega) \) is the unique solution of the Neumann problem: \( -\text{div}(\kappa \nabla \hat{u}) = f \) in \( \Omega \), \( \kappa \nabla \hat{u} \cdot n = q^D \cdot n \) on \( \Gamma \), such that \( (\mu, \hat{u} + R(q^D \cdot n - t_0) - u_0) \geq 0 \) for all \( \mu \in H^{-1/2}(\Gamma) \) with \( \mu \leq -q^D \cdot n \) on \( \Gamma_s \). Then, \((q^D, \hat{u}, \lambda)\) is a saddle point of \( \mathcal{H} \) in \( H(\text{div}; \Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \).

Proof.

(i) Let \((\hat{q}, \hat{u}, \hat{\lambda})\) be a saddle point of \( \mathcal{H} \) in \( H(\text{div}; \Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \). Then, inserting \((0, 0, 2\hat{u})\) for \((u, \lambda)\) into the left inequality of (5.43), we find
\[
\int_{\Omega} \hat{u} \text{div} \, \hat{q} \, dx + \int_{\Omega} f u \, dx + \langle \hat{q} \cdot n, \hat{\lambda} \rangle_{\Gamma_s} = 0.
\]
(5.54)

Hence, the left inequality of (5.43) reduces to
\[
\int_{\Omega} u(f + \text{div} \, \hat{q}) \, dx + \langle \hat{q} \cdot n, \lambda \rangle_{\Gamma_s} \leq 0 \quad \forall (u, \lambda) \in L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s).
\]
(5.55)
In particular, taking \( \lambda = 0 \in \tilde{H}^{1/2}(\Gamma_s) \) we get \( \text{div} \, \hat{q} = -f \) in \( \Omega \). Then (5.54) and (5.55) reduce to
\[
\langle \hat{q} \cdot n, \hat{\lambda} \rangle_{\Gamma_s} = 0 \quad \text{and} \quad \langle \hat{q} \cdot n, \lambda \rangle_{\Gamma_s} \leq 0 \quad \forall \lambda \in \tilde{H}^{1/2}(\Gamma_s).
\]
(5.56)
CHAPTER 5. MIXED FEM-BEM COUPLING WITH SIGNORINI CONTACT

The latter says that \( \hat{q} \cdot \mathbf{n} \leq 0 \) on \( \Gamma_s \), and thus \( \hat{q} \in \hat{D} \).

On the other hand, using \( \text{div} \hat{q} = -f \) in \( \Omega \) and the first equation of (5.56) in the right inequality of (5.43), we obtain

\[
\Psi(q) = \mathcal{H}(q, \hat{u}, \hat{\lambda}) \leq \mathcal{H}(q, \hat{u}, \hat{\lambda}) = \Psi(q) + \int_\Omega \hat{u} \text{div} q \, dx + \int_\Omega f \hat{u} \, dx + (q \cdot \mathbf{n}, \hat{\lambda})_{\Gamma_s} \tag{5.57}
\]

for all \( q \in H(\text{div}; \Omega) \), which yields

\[
\Psi(q) \leq \Psi(\hat{q}) \quad \forall \ q \in \hat{D},
\tag{5.58}
\]

and therefore \( \hat{q} \) is the solution of \( (\hat{P}) \).

Now, since \( \hat{q} \in \hat{D} \) is a minimizer of the quadratic functional \( \mathcal{H}(\cdot, \hat{u}, \hat{\lambda}) \), there also holds

\[
a(q, q - \hat{q}) - \langle (q - \hat{q}) \cdot \mathbf{n}, R(t_0) + u_0 \rangle + \int_\Omega \hat{u} \text{div}(q - \hat{q}) \, dx + \langle (q - \hat{q}) \cdot \mathbf{n}, \hat{\lambda} \rangle_{\Gamma_s} \geq 0 \tag{5.59}
\]

for all \( q \in H(\text{div}; \Omega) \). Then, taking \( q = \phi + \hat{q} \) with \( \phi \in [C_0^\infty(\Omega)]^n \), the boundary terms vanish and we get

\[
\int_\Omega (\kappa^{-1} \phi) \cdot \phi \, dx + \int_\Omega \hat{u} \text{div} \phi \, dx = 0 \quad \forall \ \phi \in [C_0^\infty(\Omega)]^n,
\tag{5.60}
\]

which means that \( \hat{q} = \kappa \nabla \hat{u} \) in \( \Omega \).

Then, inserting

\[
\int_\Omega \nabla \hat{u} \cdot (q - \hat{q}) \, dx + \int_\Omega \hat{u} \text{div}(q - \hat{q}) \, dx = \langle (q - \hat{q}) \cdot \mathbf{n}, \hat{u} \rangle
\]

in (5.59), we obtain

\[
\langle (q - \hat{q}) \cdot \mathbf{n}, \hat{u} - R(t_0 - \hat{q} \cdot \mathbf{n}) - u_0 \rangle + \langle (q - \hat{q}) \cdot \mathbf{n}, \hat{\lambda} \rangle_{\Gamma_s} \geq 0 \quad \forall \ q \in H(\text{div}; \Omega),
\tag{5.61}
\]

from which \( \hat{u} - R(t_0 - \hat{q} \cdot \mathbf{n}) - u_0 = 0 \) on \( \Gamma_t \), and \( \hat{u} - R(t_0 - \hat{q} \cdot \mathbf{n}) - u_0 + \hat{\lambda} = 0 \) on \( \Gamma_s \).

(ii) Let \( q^D \in \hat{D} \) be the solution of \( (\hat{P}) \), and let \( u \in H^1(\Omega)/\mathbb{R} \) be a solution of the Neumann problem: \( -\text{div}(\kappa \nabla u) = f \) in \( \Omega \), \( (\kappa \nabla u) \cdot \mathbf{n} = q^D \cdot \mathbf{n} \) on \( \Gamma \). Then, Theorems 5.3 and 5.4 imply that \( q^D = \kappa \nabla u \) in \( \Omega \).

Now, we define \( u_2 = -R(q^D \cdot \mathbf{n} - t_0), \frac{\partial u_2}{\partial \mathbf{n}} = q^D \cdot \mathbf{n} - t_0 \), such that \( (u_2 + a, \frac{\partial u_2}{\partial \mathbf{n}}) \) are the Cauchy-Data of a Laplace problem which satisfies (5.2).

Noting that \( \Psi(p) = \frac{1}{2} a(p, p) - \langle p \cdot \mathbf{n}, R(t_0) + u_0 \rangle \) for all \( p \in \hat{D} \), the solution \( q^D \in \hat{D} \) of \( (\hat{P}) \) is also characterized by the inequality

\[
a(q^D, p - q^D) \geq \langle (p - q^D) \cdot \mathbf{n}, R(t_0) + u_0 \rangle \quad \forall \ p \in \hat{D}.
\]

Therefore, defining \( \hat{D}^0 := \{ v \in H^1(\Omega) : \text{div}(\nabla v) = 0 \text{ in } \Omega, \ \nabla v \cdot \mathbf{n} \leq -q^D \cdot \mathbf{n} \text{ on } \Gamma_s \} \), we find that for all \( p = q^D + \nabla v \), with \( v \in \hat{D}^0 \), the above inequality reduces to

\[
a(q^D, \nabla v) \geq \langle \nabla v \cdot \mathbf{n}, R(t_0) + u_0 \rangle \quad \forall \ v \in \hat{D}^0,
\]

or equivalently

\[
\int_\Omega \nabla u \cdot \nabla v \, dx + \langle \nabla v \cdot \mathbf{n}, R(q^D \cdot \mathbf{n} - t_0) - u_0 \rangle \geq 0 \quad \forall \ v \in \hat{D}^0,
\]
which, using that \( \int_{\Omega} \nabla u \cdot \nabla v \, dx = \langle \nabla v \cdot n, u \rangle \), leads to

\[
\langle \nabla v \cdot n, u + R(q^D \cdot n - t_0) - u_0 \rangle \geq 0 \quad \forall v \in \tilde{D}^0.
\]

Since \( \langle \nabla v \cdot n, 1 \rangle = 0 \) for all \( v \in \tilde{D}^0 \), we observe that the inequality is satisfied up to a constant, and therefore we can force the uniqueness of \( u \in H^1(\Omega)/\mathbb{R} \), say \( \hat{u} \) from now on, by demanding that

\[
\langle \mu, \hat{u} + R(q^D \cdot n - t_0) - u_0 \rangle \geq 0 \quad \forall \mu \in H^{-1/2}(\Gamma) \quad \text{with} \quad \mu \leq -q^D \cdot n \text{ on } \Gamma_s. \quad (5.62)
\]

Also, since \(-q^D \cdot n \geq 0\), we deduce that (5.62) is valid in particular for \( \mu \leq 0 \) on \( \Gamma_s \), and therefore we must have \( \hat{u} + R(q^D \cdot n - t_0) - u_0 \leq 0 \) on \( \Gamma \) and \( \hat{u} + R(q^D \cdot n - t_0) - u_0 = 0 \) on \( \Gamma_q \). In this way, recalling that \( \nu_2 = -R(q^D \cdot n - t_0) \) on \( \Gamma \), we define

\[
\hat{\lambda} = \begin{cases} 
(u_2 + u_0 - \hat{u}) & \text{on } \Gamma_s \\
0 & \text{on } \Gamma_t
\end{cases}
\]

which, using that

\[
\langle \mu, \hat{u} + R(q^D \cdot n - t_0) - u_0 \rangle = -\langle \mu^\pm, \hat{\lambda} \rangle_{\Gamma_s} = -\langle \mu^\pm, \hat{\lambda} \rangle_{\Gamma_s},
\]

hence \( \langle q^D \cdot n, \hat{\lambda} \rangle_{\Gamma_s} = 0 \).

Next, recalling that \( \text{div} \, q^D = -f \) in \( \Omega \), we obtain

\[
\mathcal{H}(q^D, u, \lambda) = \tilde{\Psi}(q^D) + \int_{\Omega} u \text{div} \, q^D \, dx + \int_{\Omega} f \, u \, dx + \langle q^D \cdot n, \lambda \rangle_{\Gamma_s}
= \tilde{\Psi}(q^D) + \langle q^D \cdot n, \lambda \rangle_{\Gamma_s} \quad \forall u \in L^2(\Omega), \quad \forall \lambda \in \tilde{H}^{1/2}_{+}(\Gamma_s). \quad (5.63)
\]

Then, since \( \lambda \geq 0, q^D \cdot n \leq 0 \) on \( \Gamma_s \), and \( \langle q^D \cdot n, \hat{\lambda} \rangle_{\Gamma_s} = 0 \), we deduce that

\[
\mathcal{H}(q^D, u, \lambda) \leq \tilde{\Psi}(q^D) = \mathcal{H}(q^D, \hat{u}, \hat{\lambda}) \quad \forall u \in L^2(\Omega), \quad \forall \lambda \in \tilde{H}^{1/2}_{+}(\Gamma_s),
\]

which proves the first inequality of (5.43) for \( (q^D, \hat{u}, \hat{\lambda}) \).

In order to prove the second inequality of (5.43), we must show that \( \mathcal{H}(q^D, \hat{u}, \hat{\lambda}) \leq \mathcal{H}(q, \hat{u}, \hat{\lambda}) \) for all \( q \in H(\text{div}; \Omega) \). Indeed, using the definition of \( \mathcal{H} \) and integrating by parts, we have

\[
\mathcal{H}(q^D, \hat{u}, \hat{\lambda}) - \mathcal{H}(q, \hat{u}, \hat{\lambda})
= \tilde{\Psi}(q^D) - \tilde{\Psi}(q) - \int_{\Omega} \nabla \hat{u} \cdot (q^D - q) \, dx + \langle (q^D - q) \cdot n, \hat{u} \rangle + \langle (q^D - q) \cdot n, \hat{\lambda} \rangle_{\Gamma_s},
\]

which, with \( \nabla \hat{u} = q^D \), leads to

\[
\mathcal{H}(q^D, \hat{u}, \hat{\lambda}) - \mathcal{H}(q, \hat{u}, \hat{\lambda})
= -\frac{1}{2} a(q^D - q, q^D - q) - a(q^D, q - q^D) - \int_{\Omega} q^D \cdot (q^D - q) \, dx
- \langle (q^D - q) \cdot n, R(t_0 + u_0 - \hat{u}) \rangle + \langle (q^D - q) \cdot n, \hat{\lambda} \rangle_{\Gamma_s}
= -\frac{1}{2} a(q^D - q, q^D - q) - \langle (q^D - q) \cdot n, R(t_0 - q^D \cdot n) + u_0 - \hat{u} \rangle + \langle (q^D - q) \cdot n, \hat{\lambda} \rangle_{\Gamma_s}.
\]
Thus, the definition of $\lambda$ and the fact that the bilinear form $a$ is positive semi-definite, yield

$$\mathcal{H}(q^D, \hat{u}, \hat{\lambda}) - \mathcal{H}(q, \hat{u}, \hat{\lambda}) = -\frac{1}{2}a(q^D - q, q^D - q) \leq 0,$$

which finishes the proof. □

Next, we prove the continuous inf-sup condition and show uniqueness and an a priori estimate for problem $(M)$.

**Lemma 5.7 (continuous inf-sup condition)** There exists a constant $\beta > 0$ such that for all $u \in L^2(\Omega)$, $\mu \in \tilde{H}^{1/2}(\Gamma_s)$ holds

$$\sup_{q \in H(\text{div}; \Omega) \setminus \{0\}} \frac{B(q, (u, \mu))}{\|q\|_{H(\text{div}; \Omega)}} \geq \beta \|u\|_{L^2(\Omega)} + \gamma \|\mu\|_{\tilde{H}^{1/2}(\Gamma_s)}.$$  \(5.64\)

**Proof.** We investigate the following auxiliary problem posed on $\Omega$

$$\begin{aligned}
-\Delta \omega &= u \quad \text{in } \Omega, \\
\omega &= 0 \quad \text{on } \Gamma_t, \\
\frac{\partial \omega}{\partial n} &= \mu \quad \text{on } \Gamma_s.
\end{aligned}$$  \(5.65\)

and define

$$q_u = -\nabla \omega.$$

We obtain that

$$\text{div } q_u = -\Delta \omega = u \quad \text{and } q_u \cdot n|_{\Gamma_s} = -\frac{\partial \omega}{\partial n}|_{\Gamma_s} = 0,$$

which implies

$$\langle \mu, q_u \cdot n \rangle|_{\Gamma_s} = 0 \quad \forall \mu \in \tilde{H}^{1/2}(\Gamma_s).$$

Then $q_u$ belongs to $H(\text{div}; \Omega)$ and there holds

$$(\text{div } q_u, u)_{L^2(\Omega)} = \|u\|_{L^2(\Omega)}^2.$$

Using the continuity dependence result we get

$$\|q_u\|_{H(\text{div}; \Omega)}^2 = \|q_u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \leq C \|u\|_{L^2(\Omega)}^2.$$

Therefore we have

$$\sup_{q \in H(\text{div}; \Omega) \setminus \{0\}} \frac{B(q, (u, \mu))}{\|q\|_{H(\text{div}; \Omega)}} \geq \frac{B(q_u, (u, \mu))}{\|q_u\|_{H(\text{div}; \Omega)}} = \frac{\|u\|_{L^2(\Omega)}^2}{\|q_u\|_{H(\text{div}; \Omega)}} \geq \frac{1}{C} \|u\|_{L^2(\Omega)}.$$

Let $\lambda \in H^{-1/2}(\Gamma_s)$ arbitrary. We introduce the following auxiliary mixed boundary value problem

$$\begin{aligned}
\Delta \omega &= 0 \quad \text{in } \Omega, \\
\frac{\partial \omega}{\partial n} &= \lambda \quad \text{on } \Gamma_s, \\
\omega &= 0 \quad \text{on } \Gamma_t.
\end{aligned}$$  \(5.66\)

This mixed boundary value problem has a unique solution $\omega$ in $H^1(\Omega)$ with the a priori estimate

$$\|\omega\|_{H^1(\Omega)} \leq C \|\lambda\|_{H^{-1/2}(\Gamma_s)}.$$

Taking

$$q_\lambda := \nabla \omega$$
we observe that
\[ \text{div } q_\lambda = \Delta \omega = 0, \] (5.67)
which implies
\[ (\text{div } q_\lambda, v)_{L^2(\Omega)} = 0 \quad \forall v \in L^2(\Omega) \]
and \( q_\lambda \in H(\text{div}; \Omega). \) Therefore we have
\[
\sup_{q \in H(\text{div}; \Omega) \setminus \{0\}} \frac{B(q, (u, \mu))}{\|q\|_{H(\text{div}; \Omega)}} \geq \frac{B(q_\lambda, (u, \mu))}{\|q_\lambda\|_{H(\text{div}; \Omega)}} = \frac{\langle \mu, q_\lambda \cdot n \rangle_{\Gamma_s}}{\|q_\lambda\|_{H(\text{div}; \Omega)}} \geq \frac{1}{C} \frac{\langle \mu, \lambda \rangle_{H^1/2(\Gamma_s)}}{\|q_\lambda\|_{H(\text{div}; \Omega)}}.
\]
Because \( \lambda \in H^{-1/2}(\Gamma_s) \) is arbitrary, we can take the supremum. \( \square \)

**Theorem 5.6** The solution of problem \((M)\) is unique.

**Proof.** Let \((q_1, u_1, \lambda_1), (q_2, u_2, \lambda_2) \in H(\text{div}; \Omega) \times L^2(\Omega) \times \dot{H}^{1/2}_+ (\Gamma_s)\) be two solutions of the variational inequality corresponding to \((M)\). Due to equation (5.44) there holds
\[
a(q_1, q_2 - q_1) + b(q_2 - q_1, u_1) + d(q_2 - q_1, \lambda_1) = \langle r, (q_2 - q_1) \cdot n \rangle, \\
a(q_2, q_2 - q_2) + b(q_1 - q_2, u_2) + d(q_1 - q_2, \lambda_2) = \langle r, (q_1 - q_2) \cdot n \rangle.
\]
Summation gives
\[
a(q_1 - q_2, q_1 - q_2) = -b(q_1 - q_2, u_1 - u_2) - d(q_1 - q_2, \lambda_1 - \lambda_2). \tag{5.68}
\]
Due to equation (5.45) there holds
\[
b(q_1, u_i) = -(f, u_i), \quad i \in \{1, 2\}, \\
b(q_2, u_i) = -(f, u_i), \quad i \in \{1, 2\}.
\]
This gives \(b(q_1 - q_2, u_1 - u_2) = 0\).
Equation (5.46) gives
\[
d(q_1, \lambda_2 - \lambda_1) \leq 0, \\
d(q_2, \lambda_1 - \lambda_2) \leq 0,
\]
i.e., summation gives
\[
-d(q_1 - q_2, \lambda_1 - \lambda_2) \leq 0. \tag{5.69}
\]
Therefore we have \(\|q_1 - q_2\|^2 = a(q_1 - q_2, q_1 - q_2) \leq 0\), i.e., \(q_1 = q_2 =: \hat{q}\).
Equation (5.44) can be read in the form
\[
a(\hat{q}, q) + B(q, (u_1, \lambda_1)) = \langle r, q \cdot n \rangle \quad \forall q \in H(\text{div}; \Omega), \\
a(\hat{q}, q) + B(q, (u_2, \lambda_2)) = \langle r, q \cdot n \rangle \quad \forall q \in H(\text{div}; \Omega).
\]
Subtraction gives
\[
B(q, (u_1 - u_2, \lambda_1 - \lambda_2)) = 0 \quad \forall q \in H(\text{div}; \Omega). \tag{5.70}
\]
The continuous inf-sup condition, Lemma 5.7, leads to
\[
0 = \sup_{q \in H(\text{div}; \Omega)} \frac{B(q, (u_1 - u_2, \lambda_1 - \lambda_2))}{\|q\|_{H(\text{div}; \Omega)}} \geq C(\|u_1 - u_2\|_{L^2(\Omega)} + \|\lambda_1 - \lambda_2\|_{\dot{H}^{1/2}_+(\Gamma_s)}),
\]
i.e., \(u_1 = u_2\) and \(\lambda_1 = \lambda_2\). \( \square \)
An a priori estimate for the solution of problem \((M)\) is established next.
**Theorem 5.7** There exists a constant $C$, independent of $r \equiv (R(t_0) + u_0)$ and $f$, such that

$$
\| \hat{q} \|_{H^1(\Omega)} + \| \hat{u} \|_{L^2(\Omega)} + \| \hat{\lambda} \|_{\tilde{H}^{1/2}(\Gamma)} \leq C \left\{ \| r \|_{H^{1/2}(\Gamma)} + \| f \|_{L^2(\Omega)} \right\}. \tag{5.71}
$$

**Proof.** First, we note that due to the continuous inf-sup condition Lemma 5.7 (c.f. the proof of Theorem 2.1 in [3]) there exists a constant $\beta > 0$, such that for all $(u, \lambda) \in L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma)$ there holds

$$
\beta \| (u, \lambda) \|_{L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma)} \leq \sup_{q \in H(\div; \Omega)} \frac{B(q, (u, \lambda))}{\| q \|_{H(\div; \Omega)}}. \tag{5.72}
$$

On the other hand (5.51) leads to

$$
\beta \| (\hat{u}, \hat{\lambda}) \| \leq \sup_{q \in H(\div; \Omega)} \frac{B(q, (\hat{u}, \hat{\lambda}))}{\| q \|_{H(\div; \Omega)}} = \sup_{q \neq 0} \frac{\langle q \cdot n, r \rangle - a(q, q)}{\| q \|_{H(\div; \Omega)}}
\leq C \left\{ \| r \|_{H^{1/2}(\Gamma)} + \| \hat{q} \|_{H(\div; \Omega)} \right\}. \tag{5.73}
$$

Now, with (5.44)—(5.46) we obtain

$$
a(q, q) = \langle q \cdot n, r \rangle - b(q, \hat{u}) - d(q, \lambda) = \langle \hat{q} \cdot n, r \rangle + (f, \hat{u})_{L^2(\Omega)}
\leq \| r \|_{H^{1/2}(\Gamma)} \| \hat{q} \|_{H(\div; \Omega)} + \| f \|_{L^2(\Omega)} \| \hat{u} \|_{L^2(\Omega)}. \tag{5.75}
$$

Using $\div \hat{q} = -f$ and (5.73), this yields

$$
\| \hat{q} \|_{H(\div; \Omega)} \leq C \left\{ \| r \|_{H^{1/2}(\Gamma)} + \| f \|_{L^2(\Omega)} \right\}.
$$

Therefore

$$
\| \hat{q} \|_{H(\div; \Omega)} \leq C \left\{ \| r \|_{H^{1/2}(\Gamma)} + \| f \|_{L^2(\Omega)} \right\}.
$$

In the following we define an equivalent formulation of problem $(M)$ which is more suitable for discretization.

**Definition 5.8** Let the non-symmetric, but positive definite, bilinear form $b_I : (H^{1/2}(\Gamma)/\IR \times H^{-1/2}(\Gamma)) \times (H^{1/2}(\Gamma)/\IR \times H^{-1/2}(\Gamma)) \to \IR$ be given by

$$
b_I(\phi, \chi; \psi) := \langle \eta, V \chi \rangle + \langle W \phi, \psi \rangle - \langle \eta, (I + K) \phi \rangle + \langle (I + K') \chi; \psi \rangle \tag{5.74}
$$

and the bilinear form $A : (H(\div; \Omega) \times H^{1/2}(\Gamma)/\IR) \times (H(\div; \Omega) \times H^{1/2}(\Gamma)/\IR)$ be given by

$$
A(p, \phi; q, \psi) = \int_{\Omega} (\kappa^{-1} p) \cdot q \, dx + b_I(\phi, p \cdot n; \psi, q \cdot n). \tag{5.75}
$$

Then, the problem $(M_I)$ reads:

Find $\hat{q}, \hat{u}, \lambda \in H(\div; \Omega) \times H^{1/2}(\Gamma)/\IR \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma)$ such that

$$
A(\hat{q}, \hat{u}; \lambda) + B(q, (\hat{u}, \hat{\lambda})) = \langle u_0, q \cdot n \rangle + b_I(0, t_0; \phi, q \cdot n)
\\forall (q, \phi) \in H(\div; \Omega) \times H^{1/2}(\Gamma)/\IR \tag{5.76}
$$

$$
B(\hat{q}, (u - \hat{u}, \lambda - \hat{\lambda})) \leq - \int_{\Omega} f (u - \hat{u}) \, dx \quad \forall (u, \lambda) \in L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma) \tag{5.77}
$$


Lemma 5.8 The problems \((M)\) and \((M_I)\) are equivalent.

Proof. Inserting \(q = 0\) into (5.76) gives the equation
\[
b_I(\hat{\phi}, \hat{q} \cdot n - t_0; \phi, 0) = \langle W\hat{\phi}, \phi \rangle + \langle (I + K')(\hat{q} \cdot n + t_0), \phi \rangle = 0.
\]
Therefore we obtain
\[
\hat{\phi} = -W^{-1}(I + K')(\hat{q} \cdot n - t_0).
\]
Inserting this into (5.76) we obtain
\[
A(\hat{q}, \hat{\phi}; q, \phi) - b_I(0, t_0; \phi, q \cdot n)
= \int_{\Omega} \langle \kappa^{-1} \hat{q} \rangle \cdot q \, dx + b_I(\hat{\phi}, \hat{q} \cdot n + t_0, \phi, q \cdot n)
= \int_{\Omega} \langle \hat{q} \rangle \cdot q \, dx + \langle q \cdot n, V(\hat{q} \cdot n - t_0) \rangle + \langle W\hat{\phi}, \phi \rangle - \langle q \cdot n, (I + K')\hat{\phi} \rangle + \langle (I + K')(\hat{q} \cdot n - t_0), \phi \rangle
= \int_{\Omega} \langle \hat{q} \rangle \cdot q \, dx + \langle q \cdot n, V(\hat{q} \cdot n - t_0) \rangle - \langle q \cdot n, (I + K)\hat{\phi} \rangle
= \int_{\Omega} \langle \hat{q} \rangle \cdot q \, dx + \langle q \cdot n, V(\hat{q} \cdot n - t_0) \rangle + \langle q \cdot n, (I + K)W^{-1}(I + K')(\hat{q} \cdot n - t_0) \rangle
= \int_{\Omega} \langle \hat{q} \rangle \cdot q \, dx + \langle q \cdot n, R(\hat{q} \cdot n - t_0) \rangle,
\]
i.e., (5.51). The other direction of the proof follows analogously. \(\square\)

5.1.4 General numerical approximation

In this section we treat the numerical approximation for problem \((\hat{P})\) by using mixed finite elements in \(\Omega\) and boundary elements on \(\Gamma\). For simplicity we assume that \(\Gamma_I\) and \(\Gamma_s\) are polygonal (i.e., piecewise straight lines) for \(n = 2\) or piecewise hyperplanes for \(n \geq 3\).

Let \((\mathcal{T}_h)_{h \in I}\) be a family of regular triangulations \([17]\) of the domain \(\Omega\) by triangles/tetrahedrons \(T\) of diameter \(h_T\) such that \(h := \max\{h_T : T \in \mathcal{T}_h\}\). We denote by \(\rho_T\) the diameter of the inscribed circle/sphere in \(T\), and assume that there exists a constant \(\kappa > 0\) such that, for any \(h\) and for any \(T \in \mathcal{T}_h\), the inequality
\[
\frac{h_T}{\rho_T} \leq \kappa \tag{5.78}
\]
holds. Moreover we assume that there exists a constant \(C > 0\) such that for any \(h\) and for any triangle/tetrahedron \(T \in \mathcal{T}_h\), such that \(T \cap \partial \Omega\) is a whole edge/face of \(T\), there holds
\[
|T \cap \partial \Omega| \geq C h^{n-1} \tag{5.79}
\]
where \(|T \cap \partial \Omega|\) denotes the length/area of \(T \cap \partial \Omega\). This means that the family of triangulations is uniformly regular near the boundary.

We assume that all points/curves in \(\Gamma_I \cap \Gamma_s\) are vertices/edges of \(\mathcal{T}_h\) for all \(h > 0\). Then, we denote by \(E_h\) the set of all edges/faces \(e\) of \(\mathcal{T}_h\) and put \(G_h := \{e \in E_h : e \subset \Gamma\}\). Further, we let \((\tau_h)_{h \in I}\) be a family of independent regular triangulations of the boundary part \(\Gamma_s\) by line segments/triangles \(\Delta\) of diameter \(\tilde{h}_\Delta\) such that \(\tilde{h} := \max\{\tilde{h}_\Delta : \Delta \in \tau_h\}\).

Next, we take \(I \subset (0, \infty)\) with \(0 \in \tilde{I}\), and consider \((X_{h,h})_{h \in I, \tilde{h} \in I} = (L_h \times H_h \times H^{-1/2}_h \times H^{1/2}_h \times H^{1/2}_s, h)\) be a family of finite-dimensional subspaces of \(X = L^2(\Omega) \times H(\text{div}; \Omega) \times H(\text{curl}; \Omega) \times H(\text{curl}; \Omega) \times H^1(\Omega)\).
$H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)/\mathbb{R} \times \tilde{H}^{1/2}(\Gamma_s)$, subordinated to the corresponding triangulations, with $H_h^{-1/2}$ being the restriction of $H_h$ on $\Gamma_s$, and such that the following approximation property holds.

$$
\lim_{h \to 0} \inf_{(u_h, q_h, \psi_h, \phi_h, \lambda_h) \in X_h} \|(u, q, \psi, \phi, \lambda) - (u_h, q_h, \psi_h, \phi_h, \lambda_h)\| = 0
$$

(5.80)

for all $(u, q, \psi, \phi, \lambda) \in X$. In addition, we assume that the divergence of the functions in $H_h$ belong to $L_h$, that is

$$
\{\text{div } q_h : q_h \in H_h\} \subseteq L_h.
$$

(5.81)

Also, the subspaces $(L_h, H^{1/2}_{s, h})$ and $H_h$ are supposed to verify the usual discrete Babuška-Brezzi condition, which means that there exists $\beta^* > 0$ such that

$$
\inf_{(u_h, \lambda_h) \in (L_h, H^{1/2}_{s, h})} \sup_{q_h \in H_h, q_h \neq 0} \frac{B(q_h, (u_h, \lambda_h))}{\|q_h\|_{H(\text{div}; \Omega)} \|(u_h, \lambda_h)\|_{L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)}} \geq \beta^*.
$$

(5.82)

For $h \in I$, $\tilde{h} \in I$ let

$$
i_h : L_h \hookrightarrow L^2(\Omega),
$$

$$
j_h : H_h \hookrightarrow H(\text{div}; \Omega),
$$

$$
k_h : H^{-1/2}_h \hookrightarrow H^{-1/2}(\Gamma),
$$

$$
l_h : H^{1/2}_h \hookrightarrow H^{1/2}(\Gamma)/\mathbb{R},
$$

$$
m_h : H^{1/2}_{s, h} \hookrightarrow \tilde{H}^{1/2}(\Gamma_s)
$$

denote the canonical imbeddings with their duals $i^*_h, j^*_h, k^*_h, l^*_h$ and $m^*_h$.

In order to approximate $R$ we define the discrete operators

$$R_h := j_h^* \gamma^* R \gamma j_h,
$$

(5.83)

$$\tilde{R}_h := j_h^* \gamma^* V \gamma j_h + j_h^* \gamma^* (I + K) l_h (l^*_h W h^{-1} l^*_h (I + K') \gamma j_h),
$$

(5.84)

where $\gamma : H(\text{div}; \Omega) \rightarrow H^{-1/2}(\Gamma)$ is the trace operator giving the normal component of functions in $H(\text{div}; \Omega)$.

We remark that the computation of $\tilde{R}_h$ requires the numerical solution of a linear system with a symmetric positive definite matrix $W_h := l^*_h W l_h$. In general, there holds $\tilde{R}_h \neq R_h$ because $\tilde{R}_h$ is a Schur complement of discretized matrices while $R_h$ is a discretized Schur complement of operators.

Then, in order to approximate the solution of problem $(M)$, we consider the following nonconforming Galerkin scheme $(M_h)$:

**Definition 5.9** The problem $(M_h)$ is:

Find $(\tilde{q}_h, \tilde{u}_h, \tilde{\lambda}_h) \in H_h \times L_h \times H^{1/2}_{s, h}$, such that

$$a_h(\tilde{q}_h, q_h) + b(q_h, \tilde{u}_h) + d(q_h, \tilde{\lambda}_h) = \langle q_h \cdot n, r_h \rangle \quad \forall q_h \in H_h,
$$

(5.85)

$$b(\tilde{q}_h, u_h) = -\int_{\Omega} f u_h \, dx \quad \forall u_h \in L_h,
$$

(5.86)

$$d(\tilde{q}_h, \lambda_h - \tilde{\lambda}_h) \leq 0 \quad \forall \lambda_h \in H^{1/2}_{s, h},
$$

(5.87)
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where

\[
H^{1/2}_{s,+h} := \{ \mu \in H^{1/2}_{s,h} : \mu \geq 0 \},
\]

\[
a_h(p, q) = \int_\Omega (\kappa^{-1} p) \cdot q \, dx + \langle q \cdot n, \nabla_h(p \cdot n) \rangle \quad \forall p, q \in H_h,
\]

\[
b(q, u) = \int_\Omega u \, \nabla q \, dx \quad \forall (q, u) \in H_h \times L_h,
\]

\[
d(q, \lambda) = \langle q \cdot n, \lambda \rangle \quad \forall (q, \lambda) \in H_h \times H^{1/2}_{s,+h},
\]

and

\[
r_h := k_h^2 \frac{1}{2} (V + (I + K)l_t(l_t^hWl_h)^{-1}l_t^h(I + K'))t_0 + u_0.
\]

Therefore, we obtain the assertion (5.93), c.f. [19, Lemma 9].

Note, that the nonconformity of problem (5.9) arises from the bilinear form \(a_h(\cdot, \cdot)\) approximating \(a(\cdot, \cdot)\).

The following lemma provides bounds for the approximation error introduced by using a discrete Schur complement.

**Lemma 5.9** Let the symmetric operator \(\delta_{r,h}\) be defined for all \(t \in H^{-1/2}(\Gamma)\) by

\[
\delta_{r,h}(t) := \frac{1}{2} (I + K)(W^{-1} - l_t(l_t^hWl_h)^{-1}l_t^h(I + K'))t \in H^{1/2}(\Gamma).
\]

Then \(\delta_{r,h}\) is bounded and there exists \(c_0 > 0\), independent of \(h\), such that for all \(t \in H^{-1/2}(\Gamma)\) holds

\[
\|\delta_{r,h}(t)\|_{H^{1/2}(\Gamma)} \leq c_0 \inf_{\phi_h \in H^{1/2}_h} \|W^{-1}(I + K')t - \phi_h\|_{H^{1/2}(\Gamma)},
\]

(5.93)

\[
(t, \delta_{r,h}(t)) \geq 0.
\]

(5.94)

**Proof.** Since \(W : H^{1/2}(\Gamma)/\mathbb{R} \rightarrow H^{-1/2}(\Gamma)\) is positive definite and we have \(\langle (I + K')t, 1 \rangle = \langle t, (I + K)1 \rangle = 0\), there exists a unique solution \(z \in H^{1/2}(\Gamma)/\mathbb{R}\) of \(Wz = (I + K')t\) and \(z_h \in H^{1/2}_h\) of \((l_t^hWl_h)z_h = l_t^h(I + K')t\), i.e., \(z_h\) is the Galerkin approximation of \(z\) in \(H^{1/2}_h\) and we have the a priori estimates

\[
\|z\|_{H^{1/2}(\Gamma)/\mathbb{R}} \leq C_1 \|(I + K')t\|_{H^{-1/2}(\Gamma)} \quad \text{and} \quad \|l_hz_h\|_{H^{1/2}(\Gamma)/\mathbb{R}} \leq C_2 \|(I + K')t\|_{H^{-1/2}(\Gamma)}.
\]

Consequently, using the boundedness of \((I + K), (I + K')\) and the a priori estimates we obtain the boundedness of \(\delta_{r,h}\)

\[
\langle \delta_{r,h}t, s \rangle = \langle z - l_hz_h, (I + K)s \rangle \leq (C_1 + C_2) \|(I + K')t\|_{H^{-1/2}(\Gamma)} \|(I + K')s\|_{H^{-1/2}(\Gamma)} \leq C_3 \|t\|_{H^{-1/2}(\Gamma)} \|s\|_{H^{-1/2}(\Gamma)}.
\]

Using the boundedness of \((I + K)\) and the quasi-optimal error estimate we have

\[
\|\delta_{r,h}(t)\|_{H^{1/2}(\Gamma)} \leq \frac{1}{2} \|(I + K)\| \cdot \|z - l_hz_h\|_{H^{1/2}(\Gamma)} \leq c_0 \text{dist}_{H^{1/2}(\Gamma)}(W^{-1}(I + K')t, H^{1/2}_h).
\]

Therefore, we obtain the assertion (5.93), c.f. [19, Lemma 9].

Additionally, we can write for the error in the energy norm

\[
0 \leq \|z - z_h\|_W^2 = \|z\|_W^2 - \|l_hz_h\|_W^2 = t^*(I + K)(W^{-1} - l_h(l_t^hWl_h)^{-1}l_t^h)(I + K')t = \langle t, \delta_{r,h}(t) \rangle.
\]

This proves assertion (5.94). \(\Box\)

The following lemma provides an upper bound for the difference between the discrete and the continuous solutions above.
Lemma 5.10 Let \((\hat{q}, \hat{u}, \hat{\lambda})\) and \((\hat{q}_h, \hat{u}_h, \hat{\lambda}_h)\) be the solutions of problems \((M)\) and \((M_h)\), respectively. Then there exists \(c > 0\), independent of \(h\) and \(\hat{h}\), such that
\[
\|\hat{u} - \hat{u}_h\|^2_{L^2(\Omega)} + \|\hat{\lambda} - \hat{\lambda}_h\|^2_{H^{1/2}(\Gamma_s)} \leq c \left\{ \|\hat{q} - \hat{q}_h\|^2_{H(\text{div};\Omega)} + \|\hat{u} - u_h\|^2_{L^2(\Omega)} + \|\hat{\lambda} - \hat{\lambda}_h\|^2_{H^{1/2}(\Gamma_s)} + \|\hat{\delta}_{r,h}(t_0 - \hat{\cdot} \cdot n)\|^2_{H^{1/2}(\Gamma)} \right\}
\]
and
\[
\|\hat{q} - \hat{q}_h\|^2_{H(\text{div};\Omega)} \leq c \left\{ \|\hat{q} - q_h\|^2_{H(\text{div};\Omega)} + \|\hat{u} - u_h\|^2_{L^2(\Omega)} + \|\hat{\lambda} - \hat{\lambda}_h\|^2_{H^{1/2}(\Gamma_s)} + \|\hat{\delta}_{r,h}(t_0 - \hat{\cdot} \cdot n)\|^2_{H^{1/2}(\Gamma)} - d(\hat{\cdot} \cdot n, \hat{\lambda}_h - \hat{\lambda}) \right\}
\]
for all \((q_h, u_h, \lambda_h) \in H_h \times L_h \times H^{1/2}_{s,+\hat{h}}.

Proof. Given \((q_h, u_h, \lambda_h) \in H_h \times L_h \times H^{1/2}_{s,+\hat{h}}\), we observe from (5.85) and (5.44) that
\[
a(\hat{q} - \hat{q}_h, q_h) + (q_h \cdot n, (R - \tilde{R}_h)(\hat{q}_h \cdot n)) + b(q_h, \hat{u} - \hat{u}_h) + d(q_h, \hat{\lambda} - \hat{\lambda}_h) = (q_h \cdot n, r - r_h),
\]
whence
\[
b(q_h, -\hat{u}_h) + d(q_h, -\hat{\lambda}_h) = b(q_h, -\hat{u}) + d(q_h, -\hat{\lambda}) - a(\hat{q} - \hat{q}_h, q_h) + (q_h \cdot n, r - r_h - (R - \tilde{R}_h)(\hat{q}_h \cdot n)).
\]
(5.98)
Note, that we have \((q_h \cdot n, r - r_h - (R - \tilde{R}_h)(\hat{q}_h \cdot n)) = (q_h \cdot n, k^*_h \hat{\delta}_{r,h}(t_0 - k_h \hat{q}_h \cdot n)).\)
Similarly, (5.86) and (5.45) give
\[
b(\hat{q} - \hat{q}_h, u_h) = 0.
\]
(5.99)
Next, applying the discrete inf-sup condition (5.82) and making use of (5.98), we obtain
\[
\beta^\ast \|\hat{u}_h - \hat{\cdot}_h, \hat{\lambda}_h - \hat{\lambda}_h\|_{L^2(\Omega) \times H^{1/2}(\Gamma_s)} \leq \sup_{q_h \in H_h \setminus \{0\}} \frac{b(q_h, u_h - \hat{u}_h) + d(q_h, \lambda_h - \hat{\lambda}_h)}{\|q_h\|_{H(\text{div};\Omega)}}
\]
\[
= \sup_{q_h \in H_h \setminus \{0\}} \frac{b(q_h, u_h - \hat{u}) + d(q_h, \lambda_h - \hat{\lambda}) - a(\hat{q} - \hat{q}_h, q_h)(q_h \cdot n, k^*_h \delta_{r,h}(t_0 - k_h \hat{q}_h \cdot n))}{\|q_h\|_{H(\text{div};\Omega)}}
\]
\[
\leq C \left\{ \|\hat{u} - u_h\|^2_{L^2(\Omega)} + \|\hat{\lambda} - \hat{\lambda}_h\|^2_{H^{1/2}(\Gamma_s)} + \|\hat{q} - \hat{q}_h\|^2_{H(\text{div};\Omega)} + \|\delta_{r,h}(t_0 - k_h \hat{q}_h \cdot n)\|^2_{H^{1/2}(\Gamma)} \right\},
\]
which, combined with the triangle inequality applied to \((\hat{u} - \hat{u}_h, \hat{\lambda} - \hat{\lambda}_h) = (\hat{u} - u_h, \hat{\lambda} - \hat{\lambda}_h) + (u_h - \hat{u}_h, \hat{\lambda}_h - \hat{\lambda}_h)\), and the boundedness of \(\delta_{r,h}\), yield (5.95) (see also [56, Theorem 5.1’ and Remark 5.6]).

We now let \(Y := H(\text{div};\Omega) \times L^2(\Omega) \times H^{1/2}(\Gamma_s), Y_+ := H(\text{div};\Omega) \times L^2(\Omega) \times H^{1/2}(\Gamma_s)\) and define the bounded bilinear form \(A : Y \times Y \to \mathbb{R}\) by
\[
A((\hat{q}, \hat{u}, \hat{\lambda}), (q, u, \lambda)) := a(\hat{q}, q) + b(q, \hat{u}) + d(q, \hat{\lambda}) - b(\hat{q}, u) - d(\hat{q}, \lambda)
\]
for all \((\hat{q}, \hat{u}, \hat{\lambda}), (q, u, \lambda) \in Y\) and we define the linear form
\[
\Lambda(q, u, \lambda) := \langle q \cdot n, r \rangle + (f, u)_{L^2(\Omega)}
\]
for all \((q, u, \lambda) \in Y\). Let \((\tilde{q}, \tilde{u}, \tilde{\lambda})\) be the solution of problem \(\mathcal{M}\), then there holds
\[
\mathcal{A}((\tilde{q}, \tilde{u}, \tilde{\lambda}), (q - \tilde{q}, u - \tilde{u}, \lambda - \tilde{\lambda})) \geq \Lambda(q - \tilde{q}, u - \tilde{u}, \lambda - \tilde{\lambda}) \quad \forall (q, u, \lambda) \in Y_+.
\] (5.100)
Furthermore, we let \(Y_h := H_h \times L_h \times H_h^{1/2},\ Y_{h,+} := H_h \times L_h \times H_h^{1/2}\), and define the bounded bilinear form \(\mathcal{A}_h : Y_h \times Y_h \to \mathbb{R}\) by
\[
\mathcal{A}_h((\tilde{q}_h, \tilde{u}_h, \tilde{\lambda}_h), (q_h, u_h, \lambda_h)) := a_h(\tilde{q}_h, q_h) + b(q_h, \tilde{u}_h) + d(q_h, \tilde{\lambda}_h) - b(\tilde{q}_h, u_h) - d(\tilde{q}_h, \lambda_h)
\]
for all \((\tilde{q}_h, \tilde{u}_h, \tilde{\lambda}_h), (q_h, u_h, \lambda_h) \in Y_h\) and we define the linear form
\[
\mathcal{A}_h(q_h, u_h, \lambda_h) := \langle q_h \cdot \mathbf{n}, r_h \rangle + \langle f, u_h \rangle_{L^2(\Omega)}
\]
for all \((q_h, u_h, \lambda_h) \in Y_h\). Let \((\tilde{q}_h, \tilde{u}_h, \tilde{\lambda}_h)\) be the solution of problem \(\mathcal{M}_h\), then there holds
\[
\mathcal{A}_h((\tilde{q}_h, \tilde{u}_h, \tilde{\lambda}_h), (q_h - \tilde{q}_h, u_h - \tilde{u}_h, \lambda_h - \tilde{\lambda}_h)) \geq \Lambda(q_h - \tilde{q}_h, u_h - \tilde{u}_h, \lambda_h - \tilde{\lambda}_h) \quad \forall (q_h, u_h, \lambda_h) \in Y_{h,+}.
\] (5.101)

Using (5.100) and (5.101) and the Cauchy-Schwarz inequality we obtain
\[
\|q_h - \tilde{q}_h\|_{L^2(\Omega)}^2 \leq a_h(q_h - \tilde{q}_h, q_h - \tilde{q}_h)
\]
\[
\leq a_h(q_h - \tilde{q}_h, q_h - \tilde{q}_h)
\]
\[
+ \mathcal{A}_h((\tilde{q}_h, \tilde{u}_h, \tilde{\lambda}_h), (q_h - \tilde{q}_h, u_h - \tilde{u}_h, \lambda_h - \tilde{\lambda}_h)) - \Lambda_h(q_h - \tilde{q}_h, u_h - \tilde{u}_h, \lambda_h - \tilde{\lambda}_h)
\]
\[
= a_h(q_h - \tilde{q}_h, q_h - \tilde{q}_h)
\]
\[
+ \mathcal{A}_h((\tilde{q}_h, \tilde{u}_h, \tilde{\lambda}_h), (q_h - \tilde{q}_h, u_h - \tilde{u}_h, \lambda_h - \tilde{\lambda}_h)) - \Lambda_h(q_h - \tilde{q}_h, u_h - \tilde{u}_h, \lambda_h - \tilde{\lambda}_h)
\]
\[
- \mathcal{A}((\tilde{q}, \tilde{u}, \tilde{\lambda}), (q_h - \tilde{q}_h, u_h - \tilde{u}_h, \lambda_h - \tilde{\lambda}_h)) - \Lambda(q_h - \tilde{q}_h, u_h - \tilde{u}_h, \lambda_h - \tilde{\lambda}_h)
\]
\[
\leq c \|q_h - \tilde{q}_h\|_{H(\text{div};\Omega)} \|\delta_h(t_0 - k_h q_h \cdot \mathbf{n})\|_{H^{1/2}(\Gamma)} - d(\tilde{q}, \lambda_h - \tilde{\lambda}).
\] (5.102)

Now, in order to bound \(\|\text{div}(q_h - \tilde{q}_h)\|_{L^2(\Omega)}\) we observe that \(\text{div}(q_h - \tilde{q}_h) \in L_h\) and \((\text{div}(q_h - \tilde{q}_h), u_h)_{L^2(\Omega)} = 0\) for all \(u_h \in L_h\). Therefore we deduce that
\[
\|\text{div}(q_h - \tilde{q}_h)\|_{L^2(\Omega)}^2 = (\text{div}(q_h - \tilde{q}_h), \text{div}(q_h - \tilde{q}_h))_{L^2(\Omega)}
\]
\[
= (\text{div}(q_h - \tilde{q}), \text{div}(q_h - \tilde{q}))_{L^2(\Omega)} \leq \|\text{div}(q_h - \tilde{q})\|_{L^2(\Omega)} \|\text{div}(q_h - \tilde{q})\|_{L^2(\Omega)},
\]
which gives
\[
\|\text{div}(q_h - \tilde{q}_h)\|_{L^2(\Omega)} \leq \|\text{div}(q_h - \tilde{q}_h)\|_{L^2(\Omega)},
\] (5.103)
and hence,
\[
\|\tilde{q} - \tilde{q}_h\|^2_{H(\text{div};\Omega)} \leq 2 \|\tilde{q} - q_h\|^2_{H(\text{div};\Omega)} + 2 \|q_h - \tilde{q}_h\|^2_{H(\text{div};\Omega)}
\]
\[
\leq 3 \|\tilde{q} - q_h\|^2_{H(\text{div};\Omega)} + 2 \|q_h - \tilde{q}_h\|^2_{L^2(\Omega)} \quad \forall q_h \in H_h.
\] (5.104)
Finally, combining (5.102), (5.104) and (5.95), and applying the generalized Cauchy-Schwarz inequality, we arrive to (5.96).

The main result of this section is established as follows.
Theorem 5.8 Let \((\tilde{q}, \tilde{u}, \tilde{\lambda})\) and \((\tilde{q}_h, \tilde{u}_h, \tilde{\lambda}_h)\) be the solutions of problems \((M)\) and \((M_h)\), respectively. Define \(\phi := W^{-1}(I + K')(t_0 - \tilde{q} \cdot n)\). Then there exists \(c > 0\), independent of \(h\) and \(\hat{h}\), such that the following Cea type estimate holds

\[
\|\tilde{q} - \tilde{q}_h\|_{H^1(\Omega)} + \|\tilde{u} - \tilde{u}_h\|_{L^2(\Omega)} + \|\tilde{\lambda} - \tilde{\lambda}_h\|_{H^{1/2}(\Gamma_s)} \leq c \inf_{q_h \in H_h} \|\tilde{q} - q_h\|_{H^1(\Omega)} + \inf_{u_h \in L_h} \|\tilde{u} - u_h\|_{L^2(\Omega)} + \inf_{\lambda_h \in H^{1/2}_{s,+}(\Gamma_s)} \|\tilde{\lambda} - \lambda_h\|_{H^{1/2}(\Gamma_s)} + \inf_{\phi_h \in H^{1/2}_{s,+}(\Gamma)} \|\phi - \phi_h\|_{H^{1/2}(\Gamma)}.
\]

(5.105)

Proof. We apply the estimates provided in Lemma 5.10 and Lemma 5.9. First, we estimate the term \(-d(\tilde{q}, \lambda_h - \lambda)\) by \(\|\tilde{q}\|_{H^1(\Omega)}\|\lambda - \lambda_h\|_{H^{1/2}(\Gamma_s)}\) and note that \(\|\tilde{q}\|_{H^1(\Omega)}\) is bounded due to Theorem 5.7. Due to Lemma 5.9 the term \(\|\delta_r,h(t_0 - \tilde{q} \cdot n)\|_{H^{1/2}(\Gamma)}\) is bounded by \(c_0 \inf_{\phi_h \in H^{1/2}_{s,+}(\Gamma)} \|W^{-1}(I + K')(t_0 - \tilde{q} \cdot n) - \phi_h\|_{H^{1/2}(\Gamma)}\). Applying now the results in Lemma 5.10 and taking the infimum completes the proof.

Definition 5.10 The problem \((M_{h,1})\) reads:

Find \((\tilde{q}_h, \tilde{u}_h, \tilde{\lambda}_h, \tilde{\alpha}_h)\) in \(H_h \times H^{1/2}_h \times L_h \times H^{1/2}_{s,+}\) such that

\[
A(\tilde{q}_h, \phi_h; q_h, \phi_h) + B(q_h, \tilde{u}_h, \tilde{\lambda}_h) = \langle u_0, q_h \cdot n \rangle + b_l(0, t_0; \phi_h, q_h \cdot n) \quad \forall (q_h, \phi_h) \in H_h \times H^{1/2}_h \quad (5.106)
\]

\[
B(\tilde{q}_h, (u_h - \tilde{u}_h, \lambda_h - \tilde{\lambda}_h)) \leq -\int_\Omega f(u_h - \tilde{u}_h) \, dx \quad \forall (u_h, \lambda_h, \tilde{\alpha}_h) \in L_h \times H^{1/2}_{s,+} \quad (5.107)
\]

Lemma 5.11 The problems \((M_h)\) and \((M_{h,1})\) are equivalent.

Proof. Analogous to the proof of Theorem 5.8.

Remark 5.7 The formulation of problem \((M_h)\) is motivated by problem \((M_{h,1})\). We are using \((M_h)\) for further analysis, because it is simpler to analyze due to its symmetric formulation.

5.1.5 Numerical approximation

In this section we apply the general approximation theory of the last section to the following example.

Let \(T \in T_h\) an element of a triangulation of the domain \(\Omega\). For each integer \(k \geq 0\), let \(P^k(T)\) denote the space of polynomial functions of degree \(\leq k\) on \(T\). We associate with each element \(T \in T_h\) the Raviart-Thomas space [84] of index \(k\) defined by

\[
RT^k(T) = (P^k(T))^n + \bar{x}P^k(T),
\]

where \(\bar{x} = (x_1, x_2, \ldots, x_n)\) is the space variable and

\[
\bar{x}P^k(T) = \{(x_1p, x_2p, \ldots, x_np) : p \in P^k(T)\}.
\]
Now, we introduce the space
\[ RT_{-1}^k(T_h) = \{ q \in (L^2(\Omega))^n : q|_T \in RT^k(T) \quad \forall T \in T_h \}, \]
and the finite dimensional subspace of \( H(\text{div}; \Omega) \)
\[ RT_0^k(T_h) = RT_{-1}^k(T_h) \cap H(\text{div}; \Omega), \]
i.e., the normal component of \( q_h \in RT_0^k(T_h) \) will be continuous across edges, resp. faces. In the following we will use
\[
H_h := RT_0^0(T_h), \quad L_h := \{ v \in L^2(\Omega) : v|_T \in P^0(T) \quad \forall T \in T_h \}, \quad H_{s,h}^{1/2} := \{ \mu \in C^0(\Gamma_s) : \mu|_E \in P^1(E) \quad \forall E \in T_h \}, \quad H_{s,h}^{-1/2} := \{ \mu_h \in L^2(\Gamma_s) : \mu_h|_E \in P^0(E) \quad \forall E \in \mathcal{G}_h \}.
\]
Note, that \( H_{s,h}^{-1/2} \) is the space of normal traces on \( \Gamma_s \) of the functions in \( H_h \).

Our next goal is to show that the corresponding discrete inf-sup condition is satisfied.

To this end we now observe that the equilibrium interpolation operator \( E_h : [H^1(\Omega)]^n \cap H(\text{div}; \Omega) \rightarrow H_h \) (see, e.g., [84]) can also be defined from the larger space \([H^1(\Omega)]^n \cap H(\text{div}; \Omega)\) onto \( H_{s,h} \) for all \( \delta \in (0, 1) \) (see Theorem 3.1 in [2] for convex domains, which we believe from an inspection of the original proof, is also valid for general polyhedral, non-degenerated domains). In addition, as established by Theorem 3.4 in [2], the following approximation property holds
\[
\| q - E_h(q) \|_{L^2(\Omega)} \leq C h^{\delta} |q|_{H^\delta} \quad \forall q \in [H^\delta]^n \cap H(\text{div}; \Omega), \quad \forall \delta \in (0, 1). \tag{5.108}
\]

On the other hand, it is important to note that the normal trace on \( \Gamma_s \) is well defined and continuous from \([H^\delta]^n \cap H(\text{div}; \Omega)\) onto \( H^{-1/2+\delta}(\Gamma_s) \), for all \( \delta \in [0, 1) \) such that \( \delta \neq \frac{1}{2} \) (see Theorem 2.4 and the corresponding remark in [2]).

We can now prove the following lemma. For the 2d-case see [3]. Here we apply the method of proof in [3] to cover also the 3d-case.

**Lemma 5.12** There exists \( \beta_1 > 0 \), independent of \( h \) and \( \tilde{h} \), such that for all \((u_h, \mu_h) \in (L_h, H_{s,h}^{1/2})\) it holds
\[
\sup_{q_h \in H_h \setminus \{0\}} \frac{B(q_h, (u_h, \mu_h))}{\|q_h\|_{H(\text{div}; \Omega)}} \geq \beta_1 \|u_h\|_{L^2(\Omega)}.
\]

**Proof.** We adapt the analysis in the proof of the continuous inf-sup condition in Lemma 5.7 to this discrete situation. Thus, given \((u_h, \mu_h) \in (L_h, H_{s,h}^{1/2})\), we first let \( z \in H^1(\Omega) \) be the solution of
\[
\begin{cases}
-\Delta z &= u_h \quad \text{in } \Omega, \\
\frac{\partial z}{\partial n} &= 0 \quad \text{on } \Gamma_t, \\
z &= 0 \quad \text{on } \Gamma_s.
\end{cases}
\tag{5.109}
\]
Since \( u_h \in L^2(\Omega) \) and \( \Omega \) is a polygonal or polyhedral bounded domain a classical elliptic regularity result (see, e.g., [41], [98]), guarantees that \( z \in H^{1+\delta}(\Omega) \) and \( \|z\|_{H^{1+\delta}(\Omega)} \leq C\|u_h\|_{L^2(\Omega)} \), for all \( \delta \in [0, \delta_0) \), where \( \delta_0 > 0 \). Then, we consider a fixed \( \delta \in (0, \delta_0) \) such
that $\delta \neq 1/2$, define $q_a := -\nabla z \in [H^\delta(\Omega)]^n$, and observe that $\text{div } q_a = u_h$ in $\Omega$, $q_a \cdot n = 0$ on $\Gamma_s$, and therefore

$$
\|q_u\|_{H^\delta(\Omega)^n} = \|\nabla z\|_{H^\delta(\Omega)^n} \leq \|z\|_{H^{1+\delta}(\Omega)} \leq C\|u_h\|_{L^2(\Omega)}.
$$

(5.110)

Also, it is easy to see that

$$
\|q_u\|_{H(\text{div};\Omega)} \leq \left\{ \|z\|_{H^1(\Omega)}^2 + \|u_h\|_{L^2(\Omega)}^2 \right\}^{1/2} \leq C\|u_h\|_{L^2(\Omega)}.
$$

(5.111)

Now, let $P_h$ be the orthogonal projection from $L^2(\Omega)$ onto the finite element subspace $L_h$. Since $\text{div } E_h(q_u) = P_h(\text{div } q_u)$ (commuting diagram property) and $\text{div } q_u = u_h \in L_h$, it follows that $\text{div } E_h(q_u) = u_h$ in $\Omega$. Hence, using the approximation property (5.108), we deduce that

$$
\|E_h(q_u)\|_{H(\text{div};\Omega)} \leq \|q_u - E_h(q_u)\|_{H(\text{div};\Omega)} + \|q_u\|_{H(\text{div};\Omega)}
$$

$$
= \|q_u - E_h(q_u)\|_{L^2(\Omega)} + \|q_u\|_{H(\text{div};\Omega)} \leq C \|q_u\|_{H^\delta(\Omega)^n} + \|q_u\|_{H(\text{div};\Omega)}
$$

which, in virtue of the estimates (5.110), (5.111), yields

$$
\|E_h(q_u)\|_{H(\text{div};\Omega)} \leq \tilde{C}\|u_h\|_{L^2(\Omega)}.
$$

(5.112)

In addition, since $\int_{\Gamma_s} E_h(q_u) \cdot n\, ds = \int_{\Gamma_s} q_u \cdot n\, ds$ for all edges/faces $e$ of $T_h$, and $q_u \cdot n = 0$ on $\Gamma_s$, we obtain that $E_h(q_u) \cdot n = 0$ on $\Gamma_s$.

Therefore, using (5.112) we conclude that

$$
\sup_{q_h \in H_h \setminus \{0\}} \frac{B(q_h, (u_h, \mu_h))}{\|q_h\|_{H(\text{div};\Omega)}} \geq \frac{B(E_h(q_u), (u_h, \mu_h))}{\|E_h(q_u)\|_{H(\text{div};\Omega)}} \geq \frac{\|u_h\|_{L^2(\Omega)}^2}{\|E_h(q_u)\|_{H(\text{div};\Omega)}} \geq \frac{1}{C}\|u_h\|_{L^2(\Omega)},
$$

which completes the proof.

\[ \square \]

In the following we assume that $(T_h)_{h \in I}$ is uniformly regular near $\Gamma_s$, which means that there exists $C > 0$, independent of $h$, such that $\min \{h_T : T \in T_h \} \geq C h$, for all $h \in I$.

This condition yields the inverse inequality for the space $H_h^{-1/2}$ (see Theorem 3.5 in [2]), that is, for any real numbers $s$ and $t$ with $-\frac{1}{2} \leq s \leq t \leq 0$, there exists $C > 0$ such that

$$
\|\mu_h\|_{H^s(\Gamma_s)} \leq C h^{s-t} \|\mu_h\|_{H^t(\Gamma_s)} \quad \forall \mu_h \in H_h^{-1/2}.
$$

(5.113)

Then, we have the following result.

**Lemma 5.13** There exists $\beta_2 > 0$, independent of $h$ and $\tilde{h}$, such that for all $(u_h, \mu_h) \in (L_h, H_{s, h}^{1/2})$ there holds

$$
\sup_{q_h \in H_h \setminus \{0\}} \frac{B(q_h, (u_h, \mu_h))}{\|q_h\|_{H(\text{div};\Omega)}} \geq \beta_2 \sup_{\lambda_h \in H_h^{-1/2} \setminus \{0\}} \frac{\langle \lambda_h, \mu_h \rangle_{\Gamma_s}}{\|\lambda_h\|_{H^{-1/2}(\Gamma_s)}}.
$$

\[ \text{Proof.} \] We proceed similarly as in the proof of Lemma 5.12. Given $\lambda_h \in H_h^{-1/2}$ we let $z \in H^1(\Omega)$ be the solution of

$$
\begin{cases}
-\Delta z = 0 \quad \text{in } \Omega, \\
\frac{\partial z}{\partial n} = \lambda_h \quad \text{on } \Gamma_s. 
\end{cases}
$$

(5.114)
Since $H^{-1/2}_h \subseteq L^2(\Gamma_s)$, we deduce that $z \in H^{1+\delta}(\Omega)$ and $\|z\|_{H^{1+\delta}(\Omega)} \leq C\|\lambda_h\|_{H^{-1/2+\delta}(\Gamma_s)}$, for all $\delta \in [0, \delta_0]$, where $\delta_0 > 0$. Next, we take a fixed $\delta \in (0, \delta_0)$, define $q_\lambda := -\nabla z \in \{H^\delta(\Omega)\}^n$, and observe that $\text{div} \ q_\lambda = 0$ in $\Omega$, $q_\lambda \cdot n = \lambda_h$ on $\Gamma_s$, and

$$\|q_\lambda\|_{\{H^\delta(\Omega)\}^n} \leq \|z\|_{H^{1+\delta}(\Omega)} \leq C\|\lambda_h\|_{H^{-1/2+\delta}(\Gamma_s)}. \tag{5.115}$$

Also, we note that

$$\|q_\lambda\|_{H(\text{div};\Omega)} = \|q_\lambda\|_{L^2(\Omega)} \leq \|z\|_{H^1(\Omega)} \leq C\|\lambda_h\|_{H^{-1/2}(\Gamma_s)}. \tag{5.116}$$

Because of the properties of the equilibrium interpolation operator $E_h$ we deduce that $\text{div} \ E_h(q_\lambda) = \text{div} \ q_\lambda = 0$ in $\Omega$ and $E_h(q_\lambda) \cdot n = q_\lambda \cdot n = \lambda_h$ on $\Gamma_s$. Hence, using the approximation property (5.108) and the estimates (5.115), (5.116), we get

$$\|E_h(q_\lambda)\|_{H(\text{div};\Omega)} = \|E_h(q_\lambda)\|_{L^2(\Omega)} \leq \|q_\lambda - E_h(q_\lambda)\|_{L^2(\Omega)} + \|q_\lambda\|_{L^2(\Omega)} \leq C \delta_{\Omega} \|q_\lambda\|_{\{H^\delta(\Omega)\}^n} + \|q_\lambda\|_{L^2(\Omega)} \leq C \delta_{\Omega} \|\lambda_h\|_{H^{-1/2+\delta}(\Gamma_s)} + C\|\lambda_h\|_{H^{-1/2}(\Gamma_s)},$$

which, applying the inverse inequality (5.113), yields

$$\|E_h(q_\lambda)\|_{H(\text{div};\Omega)} \leq \tilde{C}\|\lambda_h\|_{H^{-1/2}(\Gamma_s)}.$$

Therefore, given $(u_h, \mu_h) \in (L_{h,s}, H^{1/2}_{s,h})$ we can write

$$\sup_{q_h \in H_h \setminus \{0\}} \frac{B(q_h, (u_h, \mu_h))}{\|q_h\|_{H(\text{div};\Omega)}} \geq \frac{B(E_h(q_\lambda), (u_h, \mu_h))}{\|E_h(q_\lambda)\|_{H(\text{div};\Omega)}} \geq \frac{\langle \lambda_h, \mu_h \rangle_{\Gamma_s}}{\|E_h(q_\lambda)\|_{H(\text{div};\Omega)}} \geq \frac{1}{\tilde{C}} \frac{\langle \lambda_h, \mu_h \rangle_{\Gamma_s}}{\|\lambda_h\|_{H^{-1/2}(\Gamma_s)}} \forall \lambda_h \in H^{-1/2}_h$$

which completes the proof. \hfill \Box

In order to continue our analysis, we have to state additional properties of the spaces $H^{1/2}_{s,h}$ and $H^{-1/2}_h$. First, we also assume that the partition $(\tau_h)$ of $\Gamma_s$ is uniformly regular, that is there exists $C > 0$ such that $\min \{h_E : E \in \tau_h\} \geq C h$. Then, noting that $H^{1/2}_{s,h} \subseteq \tilde{H}^1(\Gamma_s)$, we have the inverse inequality for the space $H^{1/2}_{s,h}$, which means that for any real numbers $s$ and $t$ with $\frac{1}{2} \leq s \leq t \leq 1$, there exists $C > 0$ such that

$$\|\mu_h\|_{H^{t}(\Gamma_s)} \leq C h^{s-t} \|\mu_h\|_{H^{s}(\Gamma_s)} \forall \mu_h \in H^{1/2}_{s,h}. \tag{5.117}$$

We recall here that, given $s \in \left[\frac{1}{2}, \frac{3}{2}\right]$, $\tilde{H}^s(\Gamma_s)$ is defined as the traces on $\Gamma_s$ of those functions $v \in H^{s+1/2}(\Omega)$ such that $v = 0$ on $\Gamma_1$. Moreover, for $\mu \in \tilde{H}^s(\Gamma_s)$, the extension to $\Gamma_1$ by zero, say $\mu^*$, is in $H^s(\Gamma)$ and the corresponding norms are equivalent (see, e.g., [59],[66]).

On the other hand, it is well known (see, e.g., [4]) that $H^{-1/2}_h$ satisfies the following approximation property: for all $s \in (-\frac{1}{2}, \frac{3}{2})$ and for all $\lambda \in H^s(\Gamma_s)$, there exists $\hat{\lambda}_h \in H^{-1/2}_h$ such that

$$\|\lambda - \hat{\lambda}_h\|_{H^{-1/2}(\Gamma_s)} \leq C h^{s+1/2} \|\lambda\|_{H^s(\Gamma_s)}. \tag{5.118}$$

Then we have the following result providing a further lower bound for the estimate from Lemma 5.13.
Lemma 5.14 There exists $C_0, \beta_3 > 0$, independent of $h$ and $\bar{h}$, such that for all $h \leq C_0 \bar{h}$ and for all $\mu_h \in H^{1/2}_{\bar{s}, \bar{h}}$ there holds

$$\sup_{\lambda_h \in H^{1/2}_{-\bar{h}, \bar{h}} \setminus \{0\}} \frac{\langle \lambda_h, \mu_h \rangle_{\Gamma_s}}{\|\lambda_h\|_{H^{-1/2}(\Gamma_s)}} \geq \|\mu_h\|_{\bar{H}^{1/2}(\Gamma_s)}.$$

Proof. Given $\mu_h \in H^{1/2}_{\bar{s}, \bar{h}}$ we let $z \in H^1(\Omega)$ be the solution of

$$
\begin{cases}
-\Delta z + z &= 0 \quad \text{in } \Omega, \\
 z &= 0 \quad \text{on } \Gamma_t, \\
 z &= \mu_{\bar{h}} \quad \text{on } \Gamma_s.
\end{cases}
$$

Since $H^{1/2}_{\bar{s}, \bar{h}} \subset \bar{H}^1(\Gamma_s)$, we deduce that $z \in H^{1+\delta}(\Omega)$ and $\|z\|_{H^{1+\delta}(\Omega)} \leq C\|\mu_h\|_{H^{1/2+\delta}(\Gamma_s)}$, for all $\delta \in [0, \delta_0]$, where $\delta_0 > 0$. Also, we observe that the normal derivative of $z$ on $\Gamma_s$ is well defined for all $\delta \in [0, \delta_0)$, and satisfies $\frac{\partial z}{\partial n} \in H^{-1/2+\delta}(\Gamma_s)$ with $\|\frac{\partial z}{\partial n}\|_{H^{-1/2+\delta}(\Gamma_s)} \leq C\|z\|_{H^{1+\delta}(\Omega)}$ (see Theorem 2.4 in [2]). In particular, for $\delta = 0$ we have $\frac{\partial z}{\partial n} \in H^{-1/2}(\Gamma_s)$, and hence

$$
\frac{\partial z}{\partial n} \cdot \mu_h \Gamma_s = \langle \frac{\partial z}{\partial n}, z \rangle_{\Gamma_s} = \|z\|_{\bar{H}^1(\Omega)}^2 \geq \tilde{C}\|\mu_h\|_{\bar{H}^{1/2}(\Gamma_s)},
$$

(5.120)

where the inequality follows from the trace theorem and the equivalence between the norms $\|\cdot\|_{H^{1/2}(\Gamma)}$ and $\|\cdot\|_{\bar{H}^{1/2}(\Gamma_s)}$.

We now consider a fixed $\delta \in (0, \delta_0)$. Then, applying the approximation property (5.118), there exists $\lambda_h \in H^{-1/2}_h$ such that

$$
\|\frac{\partial z}{\partial n} - \lambda_h\|_{H^{-1/2}(\Gamma_s)} \leq C h^\delta \|\frac{\partial z}{\partial n}\|_{H^{-1/2+\delta}(\Gamma_s)} \leq C h^\delta \|z\|_{H^{1+\delta}(\Omega)} \leq C h^\delta \|\mu_h\|_{\bar{H}^{1/2+\delta}(\Gamma_s)}
$$

which, in virtue of the inverse inequality (5.117), yields

$$
\|\frac{\partial z}{\partial n} - \lambda_h\|_{H^{-1/2}(\Gamma_s)} \leq \tilde{C} \left( \frac{h}{\bar{h}} \right)^\delta \|\mu_h\|_{\bar{H}^{1/2}(\Gamma_s)}.
$$

(5.121)

It follows that

$$
\|\lambda_h\|_{H^{-1/2}(\Gamma_s)} \leq \tilde{C} \left( \frac{h}{\bar{h}} \right)^\delta \|\mu_h\|_{\bar{H}^{1/2}(\Gamma_s)} + C \|z\|_{H^{1}(\Omega)} \leq \tilde{C} \|\mu_h\|_{\bar{H}^{1/2}(\Gamma_s)} \quad \forall h \leq \bar{h}.
$$

Therefore, we obtain that

$$
\sup_{\lambda_h \in H^{1/2}_{-\bar{h}, \bar{h}} \setminus \{0\}} \frac{\langle \lambda_h, \mu_h \rangle_{\Gamma_s}}{\|\lambda_h\|_{H^{-1/2}(\Gamma_s)}} \geq \frac{\langle \lambda_h, \mu_h \rangle_{\Gamma_s}}{\|\lambda_h\|_{H^{-1/2}(\Gamma_s)}} \geq \frac{1}{\tilde{C}} \frac{\langle \lambda_h, \mu_h \rangle_{\Gamma_s}}{\|\mu_h\|_{\bar{H}^{1/2}(\Gamma_s)}}
$$

$$
= \frac{1}{\tilde{C} \|\mu_h\|_{\bar{H}^{1/2}(\Gamma_s)}} \left\{ \langle \frac{\partial z}{\partial n}, \mu_h \rangle_{\Gamma_s} - \langle \frac{\partial z}{\partial n} - \lambda_h, \mu_h \rangle_{\Gamma_s} \right\},
$$

which, using (5.120) and (5.121), implies that

$$
\sup_{\lambda_h \in H^{1/2}_{-\bar{h}, \bar{h}} \setminus \{0\}} \frac{\langle \lambda_h, \mu_h \rangle_{\Gamma_s}}{\|\lambda_h\|_{H^{-1/2}(\Gamma_s)}} \geq \frac{1}{\tilde{C}} \left\{ \tilde{C} - \tilde{C} \left( \frac{h}{\bar{h}} \right)^\delta \right\} \|\mu_h\|_{\bar{H}^{1/2}(\Gamma_s)}.
$$
Consequently, taking in particular $C_0 := \left( \frac{C}{2c} \right)^{1/\delta}$ we conclude that for all $h \leq C_0 \bar{h}$ and for all $\mu_h \in H^{1/2}_{s,h}$ the required inequality holds with $\beta_3 := \frac{C}{2c}$, thus completing the proof.

The discrete inf-sup condition for the bilinear form $B(\cdot, \cdot)$ now reads:

**Lemma 5.15** Let $(T_h)_{h \in I}$ be uniformly regular near $\Gamma_s$, which means that there exists $C > 0$, independent of $h$, such that $\min \{ \mu_T : T \in T_h, T \cap \Gamma_s \neq \emptyset \} \geq C h$, for all $h \in I$. Furthermore, let the partition $(\tau_h)_{h \in I}$ of $\Gamma_s$ be uniformly regular, that is there exists $C > 0$ such that $\min \{ \mu_{\Delta} : \Delta \in \tau_h \} \geq \bar{C} h$. Then there exist constants $C_0, \beta > 0$, independent of $h$ and $\bar{h}$, such that for all $h \leq C_0 \bar{h}$

$$\inf_{(u_h, \lambda_h) \in L^2(\Omega) \times H^1(\Omega)} \sup_{q_h \in H_h \setminus \{0\}} \frac{B(q_h, (u_h, \lambda_h))}{\|q_h\|_{H(\text{div};\Omega)} \| (u_h, \lambda_h) \|_{L^2(\Omega) \times H^1(\Gamma_s)}} \geq \beta > 0. \quad (5.122)$$

**Proof.** Combine the lemmas 5.14, 5.14 and 5.14.

This section ends with a result on the rate of convergence of the nonconforming Galerkin scheme $(M_h)$. For this, we need the following approximation properties of the subspaces $L_h, H_h, H^{-1/2}_h, H^{1/2}_h$, and $H^1_{s,h}$, respectively (see, e.g., [4], [17], [84]):

- **(APL)h** For all $v \in H^1(\Omega)$ there exists $v_h \in L_h$ such that
  $$\|v - v_h\|_{L^2(\Omega)} \leq C h \|v\|_{H^1(\Omega)}.$$

- **(AP)h** For all $q \in [H^1(\Omega)]^n$ with $\text{div} q \in H^1(\Omega)$, there exists $q_h \in H_h$ such that
  $$\|q - q_h\|_{H(\text{div};\Omega)} \leq C h \left\{ \|q\|_{H^1(\Omega)} + \| \text{div} q\|_{H^1(\Omega)} \right\}.$$

- **(AP)h^{-1/2}** For all $\psi \in H^{1/2}(\Gamma)$, there exists $\psi_h \in H_h^{-1/2}$ such that
  $$\|\psi - \psi_h\|_{H^{-1/2}(\Gamma)} \leq C h \|\psi\|_{H^{1/2}(\Gamma)}.$$

- **(AP)h^{1/2}** For all $\phi \in H^{3/2}(\Gamma)$, there exists $\phi_h \in H_h^{1/2}$ such that
  $$\|\phi - \phi_h\|_{H^{1/2}(\Gamma)} \leq C h \|\phi\|_{H^{3/2}(\Gamma)}.$$

- **(AP)h** For all $\lambda \in H^{3/2}(\Gamma_s)$, there exists $\lambda_h \in H^{1/2}_{s,h}$ such that
  $$\|\lambda - \lambda_h\|_{H^{1/2}(\Gamma_s)} \leq C \bar{h} \|\lambda\|_{H^{3/2}(\Gamma_s)}.$$

**Theorem 5.9** Let $(\tilde{q}, \tilde{u}, \tilde{\lambda})$ and $(\bar{q}_h, \bar{u}_h, \bar{\lambda}_h)$ be the solutions of problems $(M)$ and $(M_h)$, respectively. Assume that $\tilde{q} \in [H^1(\Omega)]^n$, $\text{div} q \in H^1(\Omega)$, $\tilde{u} \in L^2(\Omega)$, $\tilde{\lambda} \in H^{3/2}(\Gamma_s)$, $\bar{\phi} := W^{-1}(I + K') (t_0 - \tilde{q} \cdot n)$, $k \in H^{3/2}(\Gamma)$.

Then the following a priori error estimate holds

$$\|\tilde{q} - \bar{q}_h\|_{H(\text{div};\Omega)} + \|\tilde{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\tilde{\lambda} - \bar{\lambda}_h\|_{H^{1/2}(\Gamma_s)}$$

$$\leq C h \left\{ \|\tilde{q}\|_{H^1(\Omega)} + \|\text{div} \tilde{q}\|_{H^1(\Omega)} + \|\tilde{u}\|_{H^1(\Omega)} + \|\bar{\phi}\|_{H^{3/2}(\Gamma)} \right\} + C \bar{h}^{1/2} \|\bar{\lambda}\|_{H^{3/2}(\Gamma_s)}$$

with a constant $C$ independent of $h$ and $\bar{h}$, with $h \leq C_0 \bar{h}$.

**Proof.** The estimates for $\tilde{q}, \tilde{u},$ and $\tilde{\lambda}$ follow directly from Theorem 5.8, the assumed regularity of the solution of $(M)$ and the approximation properties (AP)h, (APL)h, (AP)h1/2, and (AP)h1/2.

\qed
5.2 A posteriori error estimates for mixed FEM-BEM coupling

In this section we derive an a posteriori error estimator. For simplicity we restrict ourselves to \( n = 2 \) and \( \kappa = I \). We define

\[
M := H(\text{div}; \Omega) \times H^{1/2}(\Gamma)/\mathbb{R}.
\]

**Lemma 5.16** Let \( \tilde{q} := \tilde{q}_h + q^* \in H(\text{div}; \Omega) \), where \( q^* := \nabla z \) and \( z \in H^1(\Omega)/\mathbb{R} \) is the weak solution of \(-\Delta z = f + \text{div} \tilde{q}_h \) in \( \Omega \), \( \frac{\partial z}{\partial n} = 0 \) on \( \Gamma \). Then there exists \( C > 0 \), independent of \( h, \tilde{h} \), such that

\[
C\|(\tilde{q} - \tilde{q}, \hat{\phi} - \hat{\phi}_h)\|_M^2 \leq -A((\tilde{q}_h, \hat{\phi}_h), (\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h)) + (u_0, (\tilde{q} - \tilde{q}_h) \cdot n) + b_f(0, t_0; \hat{\phi} - \hat{\phi}_h, (\tilde{q} - \tilde{q}_h) \cdot n) - \langle (\tilde{q} - \tilde{q}_h) \cdot n, \tilde{\lambda}_h \rangle_{\Gamma_s} + C\|f + \text{div} \tilde{q}_h\|_{L^2(\Omega)}\|(\tilde{q} - \tilde{q}, \hat{\phi} - \hat{\phi}_h)\|_M
\]

for all \((q_h, \phi_h) \in H_h \times H^{1/2}_h\) with \( \text{div} q_h = 0 \).

**Proof.** Due to the usual continuous dependence result we have

\[
\|q^*\|_{H(\text{div}; \Omega)} \leq C\|f + \text{div} \tilde{q}_h\|_{L^2(\Omega)}.
\] (5.123)

Now, since \( \text{div} q^* = \Delta z = -f - \text{div} \tilde{q}_h \) and \( \text{div} \tilde{q} = -f \) we obtain that \( \text{div}(\tilde{q} - \tilde{q}_h) = \text{div}(\tilde{q} - \tilde{q}_h - q^*) = 0 \). It follows from the strong ellipticity of \( A \) on \( L^2(\Omega) \times H^{1/2}(\Gamma)/\mathbb{R} \) that

\[
C\|(\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h)\|_M^2 = C\|(\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h)\|_{L^2(\Omega) \times H^{1/2}(\Gamma)/\mathbb{R}}^2
\leq A((\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h), (\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h)) = A((\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h), (\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h)) - A((q^*, 0), (\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h)).
\] (5.124)

Using the continuity of \( A \) on \( M \times M \) and (5.123) we get that

\[
A((q^*, 0), (\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h)) \leq C\|q^*\|_{H(\text{div}; \Omega)}\|(\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h)\|_M \leq C\|f + \text{div} \tilde{q}_h\|_{L^2(\Omega)}\|(\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h)\|_M.
\] (5.125)

To estimate the first term on the right hand side of (5.124), we recall from (5.76) and (5.106) that

\[
A((\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h), (q_h, \phi_h)) = -B(q_h, (\dot{u} - \dot{u}_h, \dot{\lambda} - \dot{\lambda}_h)) = -\int_\Omega (u - \dot{u}_h) \text{div} q_h \, dx - \langle q_h \cdot n, \dot{\lambda} - \dot{\lambda}_h \rangle_{\Gamma_s}
\]

for all \((q_h, \phi_h) \in H_h \times H^{1/2}_h\). In particular, if \( \text{div} q_h = 0 \), then

\[
A((\tilde{q} - \tilde{q}_h, \hat{\phi} - \hat{\phi}_h), (q_h, \phi_h)) = -\langle q_h \cdot n, \dot{\lambda} - \dot{\lambda}_h \rangle_{\Gamma_s}.
\] (5.126)

But, also from (5.76) we have with \( \delta q := \tilde{q} - \tilde{q} - q_h \), that

\[
A((\tilde{q}, \hat{\phi}), (\delta q, \hat{\phi} - \hat{\phi}_h - \phi_h)) = -B(\delta q, (\dot{u}, \dot{\lambda})) + (u_0, \delta q \cdot n) + b_f(0, t_0; \hat{\phi} - \hat{\phi}_h - \phi_h, \delta q \cdot n) = -\langle \delta q \cdot n, \dot{\lambda} \rangle_{\Gamma_s} + (u_0, \delta q \cdot n) + b_f(0, t_0; \hat{\phi} - \hat{\phi}_h - \phi_h, \delta q \cdot n)
\] (5.127)
for all \((q_h, \phi_h) \in H_h \times H_h^{1/2}\) with \(\text{div} \, q_h = 0\).

Using (5.126) and (5.127) we can rewrite the first term in (5.124)

\[
\begin{align*}
A((\hat{q} - q_h, \hat{\phi} - \phi_h), (\hat{q} - q, \hat{\phi} - \phi_h)) = & \quad A((\hat{q} - q_h, \hat{\phi} - \phi_h), (\hat{q} - q - q_h, \hat{\phi} - \phi - \phi_h)) - \langle q_h \cdot n, \hat{\lambda} - \hat{\lambda}_h \rangle_{\Gamma_s} \\
= & \quad -A((\hat{q}_h, \hat{\phi}_h), (\hat{q} - q - q_h, \hat{\phi} - \phi - \phi_h)) - \langle q_h \cdot n, \hat{\lambda} - \hat{\lambda}_h \rangle_{\Gamma_s} - \langle (\hat{q} - q - q_h) \cdot n, \hat{\lambda} \rangle_{\Gamma_s} \\
& + \langle u_0, (\hat{q} - q - q_h) \cdot n \rangle + b_I(0, t_0; \hat{\phi} - \phi_h, (\hat{q} - q - q_h) \cdot n) \\
\end{align*}
\]

By further reordering of the terms and with \(\hat{q} \cdot n |_{\Gamma} = \hat{q}_h \cdot n |_{\Gamma} \) we obtain

\[
\begin{align*}
-\langle q_h \cdot n, \hat{\lambda} - \hat{\lambda}_h \rangle_{\Gamma_s} - \langle (\hat{q} - q - q_h) \cdot n, \hat{\lambda} \rangle_{\Gamma_s} \\
= & \quad -\langle (\hat{q} - q - q_h) \cdot n, \hat{\lambda}_h \rangle_{\Gamma_s} - \langle (\hat{q} - q) \cdot n, \hat{\lambda} - \hat{\lambda}_h \rangle_{\Gamma_s} \\
= & \quad -\langle (\hat{q} - q - q_h) \cdot n, \hat{\lambda}_h \rangle_{\Gamma_s} - \langle (\hat{q} - q_h) \cdot n, \hat{\lambda} - \hat{\lambda}_h \rangle_{\Gamma_s} \\
\end{align*}
\]

Due to \(\langle \hat{q} \cdot n, \hat{\lambda} \rangle_{\Gamma_s} = \langle \hat{q}_h \cdot n, \hat{\lambda}_h \rangle_{\Gamma_s} \) and \(\hat{q} \cdot n |_{\Gamma_s} \leq 0\) and \(\hat{\lambda}, \hat{\lambda}_h \geq 0\) we have

\[
\langle (\hat{q} - q_h) \cdot n, \hat{\lambda} - \hat{\lambda}_h \rangle_{\Gamma_s} \geq 0.
\]

This term is positive and will be subtracted, therefore we can bound it by zero. □

Now, let \(H^1_0\) be the usual space of all continuous piecewise linear functions corresponding to \(T_h\). Let \(E_h(\Omega)\) denote the set of all edges of the mesh \(T_h\), let \(E_h(\Gamma)\) denote the set of all edges of \(T_h|_{\Gamma}\), let \(E(T)\) denote the set of all edges of element \(T\) and let \(\tilde{E}\) denote the direction of the edges. Now, let \(I_h : H^1(\Omega) \to H^1_h\) be the interpolation operator of Clément. It is well known that \(I_h\) satisfies the following local error estimates.

**Lemma 5.17** [36] There exists positive constants \(C_1\) and \(C_2\), independent of \(h\), such that for all \(\varphi \in H^1(\Omega)\), and for any \(T \in T_h\) and \(e \in E_h\), it holds that

\[
\begin{align*}
\| \varphi - I_h \varphi \|_{L^2(T)} & \leq C_1 h_T \| \varphi \|_{H^1(\Delta(T))}, \\
\| \varphi - I_h \varphi \|_{L^2(e)} & \leq C_1 h_e^{1/2} \| \varphi \|_{H^1(\Delta(e))},
\end{align*}
\]

where \(h_e\) is the diameter of edge \(e\) and

\[
\begin{align*}
\Delta(T) := & \bigcup \{ T' \in T_h : T' \cap T \neq \emptyset \}, \\
\Delta(e) := & \bigcup \{ T' \in T_h : T' \cap e \neq \emptyset \}.
\end{align*}
\]

Next, we will prove the error estimate for the main variables \((\hat{q}_h, \hat{\phi}_h)\) following the work in [20] and [38].

**Theorem 5.10** Let \(n = 2\). There exists \(C > 0\), independent of \(h\), such that

\[
\|(\hat{q} - q_h, \hat{\phi} - \phi_h)\|_{L^2(M)} \leq C \left( \sum_{T \in T_h} \eta^2_{h,T} \right)^{1/2},
\]

where for any triangle \(T \in T_h\) we define

\[
\eta^2_{h,T} = \| f + \text{div} \, \hat{q}_h \|_{L^2(T)}^2 + h_T^2 \| \text{curl} \, (\hat{q}_h) \|_{L^2(T)}^2 \\
+ \sum_{e \in \tilde{E}(T) \cap E_h(\Omega)} h_e \| \hat{q}_h \cdot \tilde{t} \|_{L^2(e)}^2 \\
+ \sum_{e \in \tilde{E}(T) \cap E_h(\Gamma)} h_e \| \hat{q}_h \cdot \tilde{t} + \frac{d}{ds}(\xi_h - u_0) \|_{L^2(e)}^2 + \sum_{e \in \tilde{E}(T) \cap E_h(\Gamma)} h_e \| \xi_h \|_{L^2(e)}^2 \\
+ \sum_{e \in \tilde{E}(T) \cap \Gamma_s} h_e \| \frac{d}{ds} \hat{\lambda}_h \|_{L^2(e)}^2.
\]
CHAPTER 5. MIXED FEM-BEM COUPLING WITH SIGNORINI CONTACT

We use

\[ \xi_h := V(\hat{q}_h \cdot n - t_0) - (I + K)\phi_h, \]
\[ \zeta_h := W\phi_h + (I + K')(\hat{q}_h \cdot n - t_0). \]

**Proof.** Since \( q^* \cdot n = 0 \) on \( \Gamma \) and \( \langle \hat{q} \cdot n, 1 \rangle = \langle \hat{q}_h \cdot n, 1 \rangle = \langle t_0, 1 \rangle \) we have \( \langle (\hat{q} - \hat{q}_h), 1 \rangle = 0 \) for \( n = 2 \). Because \( \Omega \) is connected and we have \( \hat{q} - \hat{q}_h \in H(\text{div}; \Omega) \) and \( \text{div}(q - \hat{q}) = 0 \) there exists a stream function \( \varphi \in H^1(\Omega) \), with \( \int_\Omega \varphi \, dx = 0 \), such that

\[ \hat{q} - \hat{q} = \text{Curl} \varphi := \left( \begin{array}{c} -\frac{\partial \varphi}{\partial x_2} \\ \frac{\partial \varphi}{\partial x_1} \end{array} \right). \]

We now introduce the Clément interpolant \( \varphi_h := I_h \varphi \in H^1_h \) and choose \( q_h := \text{Curl} \varphi_h \) in Lemma 5.16. In this way, we are now interested in estimating

\[ A := -A((\hat{q}_h, \hat{\phi}_h), (\hat{q} - \hat{q} - q_h, \hat{\phi} - \hat{\phi}_h - \phi_h)) + b_I(0, t_0; \hat{\phi} - \hat{\phi}_h - \phi_h, (\hat{q} - \hat{q} - q_h) \cdot n) + \langle u_0, (\hat{q} - \hat{q} - q_h) \cdot n \rangle 
= -\int_\Omega \hat{q}_h \cdot (\hat{q} - \hat{q} - q_h) \, dx
- b_I(\hat{\phi}_h, \hat{q}_h \cdot n - t_0; \hat{\phi} - \hat{\phi}_h - \phi_h, (\hat{q} - \hat{q} - q_h) \cdot n) + \langle u_0, (\hat{q} - \hat{q} - q_h) \cdot n \rangle 
= -\int_\Omega \hat{q}_h \cdot (\hat{q} - \hat{q} - q_h) \, dx - \langle (\hat{q} - \hat{q} - q_h) \cdot n, \xi_h - u_0 \rangle + \langle \zeta_h, \hat{\phi} - \hat{\phi}_h - \phi_h \rangle 
= -\int_\Omega \hat{q}_h \cdot \text{Curl}(\varphi - \varphi_h) \, dx + \langle \frac{d}{ds}(\varphi - \varphi_h) \times q, \xi_h - u_0 \rangle + \langle \zeta_h, \hat{\phi} - \hat{\phi}_h - \phi_h \rangle. \tag{5.129} \]

using

\[ \xi_h := V(\hat{q}_h \cdot n - t_0) - (I + K)\phi_h, \]
\[ \zeta_h := W\phi_h + (I + K')(\hat{q}_h \cdot n - t_0) \]

and

\[ \text{Curl} \varphi \cdot n = \begin{bmatrix} \frac{\partial \varphi}{\partial x_2} \\ \frac{\partial \varphi}{\partial x_1} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} t_2 \\ -t_1 \end{bmatrix} = \frac{d\varphi}{ds}. \]

Using that \( \int_\Omega = \sum_{T \in T_h} \int_T \) and applying integration by parts, i.e.,

\[ \int_\omega (\psi \cdot \text{Curl} \phi + \phi \cdot \text{Curl} \psi) \, dx = \int_{\partial\omega} \phi \cdot (\psi \cdot t) \, ds, \quad \text{curl} \psi = \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2}, \]

we obtain

\[ A = -A((\hat{q}_h, \hat{\phi}_h), (\hat{q} - \hat{q} - q_h, \hat{\phi} - \hat{\phi}_h - \phi_h)) + b_I(0, t_0; \hat{\phi} - \hat{\phi}_h - \phi_h, (\hat{q} - \hat{q} - q_h) \cdot n) + \langle u_0, (\hat{q} - \hat{q} - q_h) \cdot n \rangle 
= \sum_{T \in T_h} \int_T \text{curl}(\hat{q}_h)(\varphi - \varphi_h) \, dx - \sum_{e \in \mathcal{E}(\Omega)} (\hat{q}_h \cdot \hat{t} + \frac{d}{ds}(\xi_h - u_0), \varphi - \varphi_h)_{0,e} 
+ \sum_{e \in \mathcal{E}_h(\Gamma)} \left\{ -\left( \hat{q}_h \cdot \hat{t} + \frac{d}{ds}(\xi_h - u_0), \varphi - \varphi_h \right)_{0,e} + \langle \zeta_h, \hat{\phi} - \hat{\phi}_h - \phi_h \rangle_{0,e} \right\}. \tag{5.130} \]

for all \( \phi_h \in H^{1/2}_h \), where \((\cdot, \cdot)_{0,e}\) denotes the usual \(L^2(e)\)-inner product on edge \( e \). A suitable \( \phi_h \in H^{1/2}_h \) is chosen now. We have \( \hat{\phi} - \hat{\phi}_h \in H^{1/2}(\Gamma)/\mathbb{R} \). Then, the harmonic
extension \( w \in H^1(\Omega)/\mathbb{R} \) of \( \hat{\phi} - \hat{\phi}_h \) to \( \Omega \) is given by the solution of the following boundary value problem: \( \Delta w = 0 \) in \( \Omega \) and \( w = \hat{\phi} - \hat{\phi}_h \) on \( \Gamma \). It follows that

\[
\|w\|_{H^1(\Omega)} \leq C \|\hat{\phi} - \hat{\phi}_h\|_{H^{1/2}(\Gamma)/\mathbb{R}}. \tag{5.131}
\]

Then, we define the Clément interpolant \( w_h := I_hw \) and take \( \phi_h = w_h|\Gamma \). From Lemma 5.17 we have that

\[
\|\hat{\phi} - \hat{\phi}_h - \phi_h\|_{L^2(e)} = \|w - w_h\|_{L^2(e)} \leq C h_e^{1/2} \|w\|_{H^1(\Delta(e))}.
\]

We deduce that

\[
\sum_{e \in \mathcal{E}_h(\Gamma)} (\zeta_e, \hat{\phi} - \hat{\phi}_h - \phi_h)_{0,e} \leq C \sum_{e \in \mathcal{E}_h(\Gamma)} h_e^{1/2} \|\zeta_e\|_{L^2(e)} \|w\|_{H^1(\Delta(e))}
\]

\[
\leq C \left( \sum_{e \in \mathcal{E}_h(\Gamma)} h_e \|\zeta_e\|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h(\Gamma)} \|w\|_{H^1(\Delta(e))}^2 \right)^{1/2}
\]

\[
\leq C \left( \sum_{e \in \mathcal{E}_h(\Gamma)} h_e \|\zeta_e\|_{L^2(e)}^2 \right)^{1/2} \|w\|_{H^1(\Omega)}, \tag{5.132}
\]

where the last inequality uses that the number of triangles in \( \Delta(e) \) is bounded, independently of \( h \).

For the remaining terms in (5.130), we again apply Cauchy-Schwarz’s inequality and the error estimates for the Clément interpolant \( \varphi_h \) (c.f. Lemma 5.17), and we use the fact that the numbers of triangles in \( \Delta(T) \) are also bounded independently of \( h \), to conclude that

\[
A \leq C \left( \sum_{T \in T_h} \left( h_T^2 \|\text{curl}(\hat{q}_h)\|^2_{L^2(T)} + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} h_e \|\hat{q}_h \cdot \hat{t}\|^2_{L^2(e)} \right) + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} \frac{d}{ds} (\xi_h - u_0)\|_{L^2(e)} \right)^{1/2} \|\varphi\|_{H^1(\Omega)}
\]

\[
+ C \left( \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} \frac{d}{ds} (\xi_h - u_0)\|_{L^2(e)} \right)^{1/2} \|w\|_{H^1(\Omega)}. \tag{5.133}
\]

But, since \( \varphi \) has zero mean value in \( \Omega \), we can bound

\[
\|\varphi\|_{H^1(\Omega)} \leq C \|\phi\|_{H^1(\Omega)} \leq C \|\hat{q} - \tilde{q}\|_{L^2(\Omega)}.
\]

Together with (5.131) we have

\[
|A| \leq C \left( \sum_{T \in T_h} \tilde{m}_T^2 \right)^{1/2} \left( \|\hat{q} - \tilde{q}\|_{L^2(\Omega)} + \|\hat{\phi} - \hat{\phi}_h\|_{H^{1/2}(\Gamma)/\mathbb{R}} \right)^{1/2}
\]

using

\[
\tilde{m}_T^2 := h_T^2 \|\text{curl}(\hat{q}_h)\|^2_{L^2(T)} + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} h_e \|\hat{q}_h \cdot \hat{t}\|^2_{L^2(e)}
\]

\[
+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} \frac{d}{ds} (\xi_h - u_0)\|_{L^2(e)} + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \|\zeta_{e}\|_{L^2(e)}.
\]
Due to Lemma 5.16 there remains the term \( \langle (\hat{q} - \bar{q} - q_h) \cdot \mathbf{n}, \hat{\lambda}_h \rangle_{\Gamma_s} \) to be estimated. We have

\[
\langle (\hat{q} - \bar{q} - q_h) \cdot \mathbf{n}, \hat{\lambda}_h \rangle_{\Gamma_s} = \langle \text{Curl}(\varphi - \varphi_h), \hat{\lambda}_h \rangle_{\Gamma_s} = -\langle \frac{d}{ds}(\varphi - \varphi_h), \hat{\lambda}_h \rangle_{\Gamma_s} = \int_{\Gamma_s} (\varphi - \varphi_h) \frac{d}{ds} \hat{\lambda}_h \, ds \\
\le \sum_{T \in T_h} \sum_{e \in E(T) \cap \Gamma_s} \|\varphi - \varphi_h\|_{L^2(e)} \|\frac{d}{ds} \hat{\lambda}_h\|_{L^2(e)} \\
\le C \sum_{T \in T_h} \sum_{e \in E(T) \cap \Gamma_s} h_e^{1/2} \|\hat{\lambda}_h\|_{L^2(e)} \|\varphi\|_{H^1(\Delta(e))}.
\]

Combining all terms, we obtain

\[
\|(\hat{q} - \bar{q}_h, \hat{\phi} - \bar{\phi}_h)\|_M \le C \left( \sum_{T \in T_h} \eta_{T}^2 \right)^{1/2}.
\]

\[\square\]

**Theorem 5.11** There exists \( C > 0 \), independent of \( h, \tilde{h} \), such that

\[
\|(\hat{u} - \bar{u}_h, \hat{\lambda} - \bar{\lambda}_h)\|_{L^2(\Omega) \times \bar{H}^{1/2}(\Gamma_s)} \le C \left( \sum_{T \in T_h} \eta_{T}^2 \right)^{1/2}
\]

with

\[
\eta_{T}^2 = \eta_{T}^2 + h_T^2 \|\bar{q}_h\|_{L^2(T)}^2 + \sum_{e \in E(T) \cap \Gamma} h_e \|\xi_h - u_0 + \check{\lambda}_h - s_h\|_{L^2(e)}^2,
\]

where \( s_h \) is the \( L^2 \)-projection of \( \xi_h - u_0 + \check{\lambda}_h \) onto the space of piecewise constant functions on \( T_h \cap \Gamma \).

**Proof.** The continuous inf-sup condition satisfied by the bilinear form \( B(\cdot, (\cdot, \cdot)) \) implies that

\[
\|\hat{\lambda} - \bar{\lambda}_h\|_{\bar{H}^{1/2}(\Gamma_s)} \le C \sup_{q \in H(\text{div}; \Omega) \setminus \{0\}} \frac{\langle \hat{\lambda} - \bar{\lambda}_h, q \cdot \mathbf{n} \rangle_{\Gamma_s}}{\|q\|_{H(\text{div}; \Omega)}}.
\]

Using (5.76) we can write for any \( q \in H(\text{div}; \Omega) \) with \( \text{div} \, q = 0 \)

\[
\langle \hat{\lambda}, q \cdot \mathbf{n} \rangle_{\Gamma_s} = -A(\hat{q}, \hat{\phi}; q, 0) + \langle u_0, q \cdot \mathbf{n} \rangle + b_I(0, 0; 0, q \cdot \mathbf{n}).
\]

Since \( \Omega \) is connected, there exists \( \varphi \in H^1(\Omega) \) such that \( \int_\Omega \varphi \, dx = 0 \) and \( q = \text{Curl} \, \varphi \).

Next, we let \( \varphi_h \) be the Clément interpolant of \( \varphi \) and define \( q_h := \text{Curl} \, \varphi_h \in H_h \), we also have \( \text{div} \, q_h = 0 \), and hence, (5.106) gives

\[
\langle \hat{\lambda}_h, q_h \cdot \mathbf{n} \rangle_{\Gamma_s} = -A(\hat{q}_h, \hat{\phi}_h; q_h, 0) + \langle u_0, q_h \cdot \mathbf{n} \rangle + b_I(0, t_0; 0, q_h \cdot \mathbf{n}).
\]
Therefore, we have
\[
\langle \hat{\lambda} - \hat{\lambda}_h, q \cdot n \rangle_{\Gamma_s} = \langle \hat{\lambda}, q \cdot n \rangle_{\Gamma_s} - \langle \hat{\lambda}_h, (q - q_h) \cdot n \rangle_{\Gamma_s} - \langle \hat{\lambda}_h, q_h \cdot n \rangle_{\Gamma_s}
\]
\[
= -A(\hat{q}, \hat{\phi}; q, 0) + \langle u_0, q \cdot n \rangle + b_I(0, t_0; 0, q \cdot n) - (\hat{\lambda}_h, (q - q_h) \cdot n)_{\Gamma_s}
\]
\[
- (\hat{\lambda}_h, (q - q_h) \cdot n)_{\Gamma_s}
\]
\[
= -A(\hat{q}_h, \hat{\phi}_h; q, 0) + \langle u_0, q_h \cdot n \rangle + b_I(0, t_0; 0, q_h \cdot n)
\]
\[
- A(\hat{q}_h, \hat{\phi}_h; q - q_h, 0) - (\hat{\lambda}_h, (q - q_h) \cdot n)_{\Gamma_s}
\]
\[
- \int_{\Omega} \hat{q}_h \cdot (q - q_h) \, dx + \langle u_0, (q - q_h) \cdot n \rangle - b_I(\hat{\phi}_h, \hat{q}_h \cdot n - t_0; 0, (q - q_h) \cdot n)
\]
\[
= -A(\hat{q} - \hat{q}_h, \hat{\phi} - \hat{\phi}_h; q, 0)
\]
\[
- \int_{\Omega} \hat{q}_h \cdot (q - q_h) \, dx - (\xi_h - u_0, (q - q_h) \cdot n) - (\hat{\lambda}_h, (q - q_h) \cdot n)_{\Gamma_s}.
\]

Due to the continuity of $A$ we have
\[
| - A(\hat{q} - \hat{q}_h, \hat{\phi} - \hat{\phi}_h; q, 0)| \leq C \| (\hat{q} - \hat{q}_h, \hat{\phi} - \hat{\phi}_h) \|_{M} \| q \|_{H(\text{div}; \Omega)}
\]
\[
\leq C \left( \sum_{T \in \mathcal{T}_h} \eta_{T, T}^2 \right)^{1/2} \| q \|_{H(\text{div}; \Omega)}.
\]

Note, that we can write $(q - q_h) \cdot n = \text{Curl}(\varphi - \varphi_h) \cdot n = -\frac{d}{ds}(\varphi - \varphi_h)$. Hence, using integration by parts we obtain
\[
- \int_{\Omega} \hat{q}_h \cdot (q - q_h) \, dx - (\xi_h - u_0, (q - q_h) \cdot n) - (\hat{\lambda}_h, (q - q_h) \cdot n)_{\Gamma_s}
\]
\[
= - \int_{\Omega} \hat{q}_h \cdot \text{Curl}(\varphi - \varphi_h) \, dx - (\xi_h - u_0 + \hat{\lambda}_h, \frac{d}{ds}(\varphi - \varphi_h))_{\Gamma_s}
\]
\[
= \sum_{T \in \mathcal{T}_h} \int_{T} \text{curl}(\hat{q}_h) \cdot (\varphi - \varphi_h) \, dx - \sum_{e \in \mathcal{E}_h(\Omega)} ([\hat{q}_h \cdot \hat{t}]_e, \varphi - \varphi_h)_{0, e}
\]
\[
- \sum_{e \in \mathcal{E}_h(\Gamma)} (\hat{q}_h \cdot \hat{t} + \frac{d}{ds}(\xi_h - u_0 + \hat{\lambda}_h), \varphi - \varphi_h)_{0, e}.
\]

Repeating now the arguments in the proof of Theorem 5.10 we obtain
\[
\| \hat{\lambda} - \hat{\lambda}_h \|_{H^{1/2}(\Gamma_s)} \leq C \left( \sum_{T \in \mathcal{T}_h} \eta_{T, T}^2 \right)^{1/2}.
\]

The continuous inf-sup condition satisfied by the bilinear form $B(\cdot, (\cdot, \cdot))$ implies that
\[
\| \hat{u} - \hat{u}_h \|_{L^2(\Omega)} \leq C \sup_{q \in H(\text{div}; \Omega)} \frac{\int_{\Omega} (\hat{u} - \hat{u}_h) \, \text{div} \, q \, dx}{\| q \|_{H(\text{div}; \Omega)}}.
\]  \hspace{1cm} (5.135)

In the following we have to use the equilibrium interpolation operator to construct a function in $H_h$. The application of this operator and some error estimates demand a higher regularity than $H(\text{div}; \Omega)$.
Let $q \in H(\text{div}; \Omega)$ be arbitrary but fixed. Let $\Omega' \supset \Omega$ be convex. Let $z \in H^1(\Omega')$ be the unique solution of
\[
\Delta z = \left\{ \begin{array}{ll}
\text{div} \frac{q}{\text{meas}(\Omega|\Omega)} & \text{in } \Omega \\
\text{div} q & \text{in } \Omega' \setminus \Omega \\
\frac{\partial z}{\partial n} = 0 & \text{on } \partial \Omega', \end{array} \right.
\int_{\Omega'} z \, dx = 0.
\]
A classical regularity result implies that $z \in H^2(\Omega')$ and
\[
\|z\|_{H^2(\Omega')} \leq C(1 + \frac{1}{\text{meas}(\Omega|\Omega)})^{1/2} \|\text{div} q\|_{L^2(\Omega)}.
\]
Defining $r_q := \nabla z|_{\Omega}$ we observe that $r_q \in [H^1(\Omega)]^n$, $\text{div} r_q = \text{div} q$ in $\Omega$ and
\[
\|r_q\|_{[H^1(\Omega)]^n} \leq \|z\|_{H^2(\Omega')} \leq \|z\|_{H^2(\Omega')} \leq C\|\text{div} q\|_{L^2(\Omega)}.
\]
Now, for $q \in H(\text{div}; \Omega)$, we can use $r_q$ and the properties of the equilibrium interpolation operator $E_h$ and obtain
\[
\int_{\Omega} (\hat{u} - \hat{u}_h) \, \text{div} q \, dx = \int_{\Omega} (\hat{u} - \hat{u}_h) \, \text{div} q \, dx = \int_{\Omega} \hat{u} \, \text{div} q \, dx - \int_{\Omega} \hat{u}_h \, \text{div} E_h r_q \, dx.
\]
Using (5.76) and (5.106) we can write
\[
\int_{\Omega} \hat{u} \, \text{div} r_q \, dx = -A(\hat{q}, \hat{\phi}; r_q; 0) + \langle r_q \cdot n, u_0 \rangle + b_I(0, t_0; 0, r_q \cdot n) - \langle r_q \cdot n, \lambda \rangle_{\Gamma_s} \tag{5.136}
\]
and
\[
\int_{\Omega} \hat{u}_h \, \text{div} E_h r_q \, dx = -A(\hat{q}_h, \hat{\phi}_h; E_h r_q; 0) + \langle E_h r_q \cdot n, u_0 \rangle + b_I(0, t_0; 0, E_h r_q \cdot n) - \langle E_h r_q \cdot n, \lambda_h \rangle_{\Gamma_s} \tag{5.137}
\]
which yield
\[
\int_{\Omega} (\hat{u} - \hat{u}_h) \, \text{div} q \, dx = -A(\hat{q} - \hat{q}_h, \hat{\phi} - \hat{\phi}_h; r_q; 0) + A(\hat{q}_h, \hat{\phi}_h; E_h r_q - r_q, 0)
\]
\[
+ \langle (r_q - E_h r_q) \cdot n, u_0 \rangle + b_I(0, t_0; 0, (r_q - E_h r_q) \cdot n)
\]
\[
- \langle (r_q - E_h r_q) \cdot n, \lambda_h \rangle_{\Gamma_s} - \langle r_q \cdot n, \lambda_h \rangle_{\Gamma_s}.
\]
The boundedness of $A$ provides
\[
| - A(\hat{q} - \hat{q}_h, \hat{\phi} - \hat{\phi}_h; r_q, 0) - \langle r_q \cdot n, \lambda_h \rangle_{\Gamma_s} |
\]
\[
\leq C\| \hat{q} - \hat{q}_h \|_H^2 + \| \lambda_h \|_{H^1/2(\Gamma_s)}^2 \| r_q \|_{H(\text{div}; \Omega)} \tag{5.138}
\]
We also have
\[
A(\hat{q}_h, \hat{\phi}_h; E_h r_q - r_q, 0) + b_I(0, t_0; 0, (r_q - E_h r_q) \cdot n)
\]
\[
+ \langle (r_q - E_h r_q) \cdot n, u_0 \rangle - \langle (r_q - E_h r_q) \cdot n, \lambda_h \rangle_{\Gamma_s}
\]
\[
= \int_{\Omega} \hat{q}_h \cdot (E_h r_q - r_q) \, dx + b_I(\hat{\phi}_h \hat{q}_h \cdot n - t_0; 0, (E_h r_q - r_q) \cdot n) + \langle (r_q - E_h r_q) \cdot n, u_0 - \lambda_h \rangle_{\Gamma_s}
\]
\[
= \int_{\Omega} \hat{q}_h \cdot (E_h r_q - r_q) \, dx + \langle (E_h r_q - r_q) \cdot n, \xi_h - u_0 + \lambda_h \rangle.
\]
Using the approximation results for the equilibrium interpolation operator we can estimate

\[
\int \Omega \hat{q}_h \cdot (E_h r_q - r_q) \, dx = \sum_{T \in T_h} \int_T \hat{q}_h \cdot (E_h r_q - r_q) \, dx
\]

\[
\leq C \sum_{T \in T_h} \| \hat{q}_h \|_{L^2(T)} h_T \| r_q \|_{H^1(T)}^n
\]

\[
\leq C \left\{ \sum_{T \in T_h} h_T^2 \| \hat{q}_h \|^2_{L^2(T)} \right\}^{1/2} \| q \|_{H(\text{div}; \Omega)}. \tag{5.139}
\]

\(E_h\) satisfies \(\int_e E_h(r_q) \cdot n \, ds = \int_e r_q \cdot n \, ds\) for all \(e \in \mathcal{E}_h(\Gamma)\). Therefore we can introduce \(S_h\), the space of piecewise constant functions on \(\mathcal{E}_h(\Gamma)\) induced by \(T_h\), and \(s_h\) to be the \(L^2\)-projection of \(\xi_h - u_0 + \lambda_h\) onto \(S_h\). Hence, we can write

\[
|\langle (E_h r_q - r_q) \cdot n, \xi_h - u_0 + \lambda_h \rangle | = |\langle (E_h r_q - r_q) \cdot n, \xi_h - u_0 + \lambda_h - s_h \rangle |
\]

\[
\leq \sum_{e \in \mathcal{E}_h(\Gamma)} \| E_h r_q - r_q \|_2 \| n \|_2 \| \xi_h - u_0 + \lambda_h - s_h \|_2
\]

\[
\leq C \sum_{e \in \mathcal{E}_h(\Gamma)} h_e^{1/2} \| r_q \|_{H^1(T_e)}^n \| \xi_h - u_0 + \lambda_h - s_h \|_2
\]

\[
\leq C \left\{ \sum_{e \in \mathcal{E}_h(\Gamma)} h_e \| \xi_h - u_0 + \lambda_h - s_h \|^2_{L^2(e)} \right\}^{1/2} \| q \|_{H(\text{div}; \Omega)}. \tag{5.140}
\]

Combining (5.138), (5.139) and (5.140) we obtain the upper bound for \(\hat{u} - \hat{u}_h\):

\[
\| \hat{u} - \hat{u}_h \|^2_{L^2(\Omega)} \leq \| \lambda \|_{H^{1/2}(\Gamma_s)}^2 + C \sum_{T \in T_h} \eta_T^2
\]

\[
+ C \sum_{T \in T_h} h_T^2 \| \hat{q}_h \|^2_{L^2(T)} + C \sum_{e \in \mathcal{E}_h(\Gamma)} h_e \| \xi_h - u_0 + \lambda_h - s_h \|^2_{L^2(e)}.
\]

\(\square\)

### 5.3 Solvers for FEM-BEM dual-mixed coupling problems

For solving the discretized saddle point problem we propose a modified Uzawa algorithm that utilizes the equation for the Lagrange multiplier \(u\).

Let \(P_h : H^{1/2} \big|_{s,h} \rightarrow H^{1/2} \big|_{s+,h}\) be the projection onto \(H^{1/2} \big|_{s+,h}\), which is uniquely defined by

\[
(P_h \lambda - \lambda_h, \lambda_h - P_h \lambda)_{\tilde{H}^{1/2}(\Gamma_s)} \geq 0 \quad \forall \lambda_h \in H^{1/2} \big|_{s+,h}. \tag{5.141}
\]

Let the mapping \(\Phi(q) : H(\text{div}; \Omega) \rightarrow \tilde{H}^{1/2} \big|_{s,h} \subset \tilde{H}^{1/2}(\Gamma_s)\) be defined by

\[
d(q, \lambda_h) = (\lambda_h, \Phi(q))_{\tilde{H}^{1/2}(\Gamma_s)} \quad \forall \lambda_h \in H^{1/2} \big|_{s,h}. \tag{5.142}
\]

Therefore we have

\[
\| \Phi(q) \|^2_{\tilde{H}^{1/2}(\Gamma_s)} = (\Phi(q), \Phi(q))_{\tilde{H}^{1/2}(\Gamma_s)} = d(q, \Phi(q)) = \langle q \cdot n, \Phi(q) \rangle_{\Gamma_s} \]

\[
\leq \| q \cdot n \|_{H^{-1/2}(\Gamma_s)} \| \Phi(q) \|_{\tilde{H}^{1/2}(\Gamma_s)}. \]

For solving the discretized saddle point problem we propose a modified Uzawa algorithm
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which yields

$$\|\Phi(q)\|_{H^{1/2}(\Gamma_s)} \leq \|q \cdot n\|_{H^{-1/2}(\Gamma_s)} \leq \|q\|_{H(\text{div};\Omega)}$$

proving the continuity of the linear mapping $\Phi$.

The Uzawa-algorithm reads

**Algorithm 5.1 (Uzawa(mixed))**

1. Choose an initial $\lambda(0) \in H^{1/2}_{s,+}$. 
2. Given $\lambda^{(n)} \in H^{1/2}_{s,+}$, Find $q^{(n)}$, $u^{(n)} \in H_h \times L_h$ which satisfy

   $$a_h(q^{(n)}, q_h) + b(q_h, u^{(n)}) = \langle r_h, q_h \cdot n \rangle - d(q_h, \lambda^{(n)}) \quad \forall q_h \in H_h$$
   (5.143)

   $$b(q^{(n)}, u_h) = -(f, u_h)_0 \quad \forall u_h \in L_h$$
   (5.144)

3. Compute $\lambda^{(n+1)}$ by

   $$\lambda^{(n+1)} = P_h(\lambda^{(n)} + \varrho \Phi(q^{(n)}))$$
   (5.145)

4. Check for some stopping criterion, if it is not fulfilled goto step 2

The theorem now reads

**Theorem 5.12** The modified Uzawa algorithm converges for any initial value $\lambda(0) \in H^{1/2}_{s,+}$ towards the solution of the discrete problem $(M_h)$ for $0 < \varrho < 2$.

**Proof.** Let $\hat{q}_h, \hat{u}_h, \hat{\lambda}_h \in H_h \times L_h \times H^{1/2}_{s,+}$ be the solution of problem $(M_h)$. Due to (5.87) there holds

$$d(\hat{q}_h, \lambda_h - \hat{\lambda}_h) \leq 0 \quad \forall \lambda_h \in H^{1/2}_{s,+},$$

i.e., there holds for all $\varrho > 0$

$$(\lambda_h - \hat{\lambda}_h, \lambda_h - (\hat{\lambda}_h + \varrho \Phi(\hat{q}_h)))_{H^{1/2}(\Gamma_s)} \geq 0 \quad \forall \lambda_h \in H^{1/2}_{s,+}. $$

Therefore we have

$$\hat{\lambda}_h = P_h(\hat{\lambda}_h + \varrho \Phi(\hat{q}_h)).$$

(5.146)

Now we obtain

$$\|\lambda^{(n+1)} - \hat{\lambda}_h\|_{H^{1/2}(\Gamma_s)}^2 = \|P_h(\lambda^{(n)} + \varrho \Phi(q^{(n)})) - P_h(\lambda^{(n)} + \varrho \Phi(q^{(n)}))\|_{H^{1/2}(\Gamma_s)}^2$$

$$\leq \|\lambda^{(n)} - \hat{\lambda}_h\|_{H^{1/2}(\Gamma_s)}^2 + 2\varrho \langle \lambda^{(n)} - \hat{\lambda}_h, \Phi(q^{(n)} - \Phi(\hat{q}_h)) \rangle_{H^{1/2}(\Gamma_s)} + \varrho^2 \|\Phi(q^{(n)} - \Phi(\hat{q}_h))\|_{H^{1/2}(\Gamma_s)}^2$$

$$= \|\lambda^{(n)} - \hat{\lambda}_h\|_{H^{1/2}(\Gamma_s)}^2 + 2\varrho d(q^{(n)} - \hat{q}_h, \lambda^{(n)} - \hat{\lambda}_h) + \varrho^2 \|\Phi(q^{(n)} - \Phi(\hat{q}_h))\|_{H^{1/2}(\Gamma_s)}^2.$$  

Due to (5.85) and (5.143) we have

$$d(q^{(n)} - \hat{q}_h, \lambda^{(n)} - \hat{\lambda}_h) = -a_h(q^{(n)} - \hat{q}_h, q^{(n)} - \hat{q}_h).$$

Due to (5.86) and (5.144) we have for $q^{(n)} - \hat{q}_h \in H_h$ that

$$(\text{div}(q^{(n)} - \hat{q}_h), u_h)_0 = 0 \quad \forall u_h \in L_h,$$
Applying the discrete inf-sup condition we obtain
\[ d(q^{(n)} - \hat{q}_h, \lambda^{(n)} - \hat{\lambda}_h) = -a_h(q^{(n)} - \hat{q}_h, q^{(n)} - \hat{q}_h) \leq -\|q^{(n)} - \hat{q}_h\|_{L^2(\Omega)}^2 = -\|q^{(n)} - \hat{q}_h\|_{H(\text{div};\Omega)}^2. \]

Together with \( \|\Phi(q_h)\|_{H^{1/2}(\Gamma_s)} \leq \|q_h\|_{H(\text{div};\Omega)} \) we obtain
\[ \|\lambda^{(n+1)} - \hat{\lambda}_h\|_{H^{1/2}(\Gamma_s)}^2 \leq \|\lambda^{(n)} - \hat{\lambda}_h\|_{H^{1/2}(\Gamma_s)}^2 - \varrho(2 - \varrho)\|q^{(n)} - \hat{q}_h\|_{H(\text{div};\Omega)}^2. \]  

(5.147)

Now, recalling standard arguments, we have for \( 0 < \varrho < 2 \), i.e., \( 2 - \varrho > 0 \), that \( \|\lambda^{(n)} - \hat{\lambda}_h\|_{H^{1/2}(\Gamma_s)}^2 \) is monotonely decreasing and bounded from below, therefore convergent. By

\[ 0 \leq \varrho(2 - \varrho)\|q^{(n)} - \hat{q}_h\|_{H(\text{div};\Omega)}^2 \leq \|\lambda^{(n)} - \hat{\lambda}_h\|_{H^{1/2}(\Gamma_s)}^2 - \|\lambda^{(n+1)} - \hat{\lambda}_h\|_{H^{1/2}(\Gamma_s)}^2 \to 0 \quad (n \to \infty) \]

we have
\[ \lim_{n \to \infty} \|q^{(n)} - \hat{q}_h\|_{H(\text{div};\Omega)}^2 = 0. \]

(5.148)

From (5.85) and (5.143) we have
\[ B(q_h, (u^{(n)} - \hat{u}_h, \lambda^{(n)} - \hat{\lambda}_h)) = a_h(\hat{q}_h - q^{(n)}), \forall q_h \in H_h. \]

Applying the discrete inf-sup condition we obtain
\[ \beta(\|u^{(n)} - \hat{u}_h\|_{L^2(\Omega)} + \|\lambda^{(n)} - \hat{\lambda}_h\|_{H^{1/2}(\Gamma_s)}) \leq \sup_{q_h \in H_h \setminus \{0\}} \frac{B(q_h, (u^{(n)} - \hat{u}_h, \lambda^{(n)} - \hat{\lambda}_h))}{\|q_h\|_{H(\text{div};\Omega)}} \]
\[ = \sup_{q_h \in H_h \setminus \{0\}} \frac{a_h(\hat{q}_h - q^{(n)}), q_h)}{\|q_h\|_{H(\text{div};\Omega)}} \]
\[ \leq C\|\hat{q}_h - q^{(n)}\|_{H(\text{div};\Omega)} \]

and the convergence of \( u^{(n)}, \lambda^{(n)} \) follows. \( \square \)

Now we propose an algorithm, which is better from a numerical point of view, because it completely avoids the computation of a Schur-complement.

**Algorithm 5.2 (Uzawa(mixed), 2nd formulation)**

1. Choose an initial \( \lambda^{(0)} \in H^{1/2}_{s+h} \).
2. Given \( \lambda^{(n)} \in H^{1/2}_{s+h} \). Find \( q^{(n)}, \phi^{(n)}, u^{(n)} \in H_h \times H^{1/2}_h \times L_h \) which satisfy

\[ A(q^{(n)}, \phi^{(n)}; q_h, \phi_h) + b(q_h, u^{(n)}) = \langle u_0, q_h \cdot n \rangle + b_I(0, t_0; \phi_h, q_h \cdot n) - d(q_h, \lambda^{(n)}) \quad \forall (q_h, \phi_h) \in H_h \times H^{1/2}_h \]

(5.149)

\[ b(q^{(n)}, u_h) = -(f, u_h), \forall u_h \in L_h. \]

(5.150)

3. Compute \( \lambda^{(n+1)} \) by

\[ \lambda^{(n+1)} = P_h(\lambda^{(n)} + \varrho \Phi(q^{(n)})). \]

(5.151)

4. Check for some stopping criterion, if it is not fulfilled goto step 2.
Remark 5.8 The equivalence to the first algorithm follows directly by the equivalence of problems \((M_h)\) and \((M_{h,1})\) and the fact, that introducing the variable \(\phi_h\) does not affect the projection step for \(\lambda_h^{(n)}\).

Remark 5.9 This algorithm not only avoids the Schur complement, previously necessary for the computation of \(R_h\), but it is also possible to precondition the solution scheme for the linear equation involved, see, e.g., [38], [77].

Remark 5.10 The operators \(P_h\) and \(\Phi\) are defined with respect to the scalar product of \(H^{1/2}(\Gamma_s)\), which is not practical from the computational point of view. Fortunately, inspection of the proof of Theorem 5.12 shows, that it is sufficient that the norm induced by the scalar product used in the algorithm is equivalent to the \(H^{1/2}(\Gamma_s)\)-norm. Therefore we can use the bilinear form \(\langle \cdot, \cdot \rangle_W\) instead of the scalar product \(\langle \cdot, \cdot \rangle_{H^{1/2}(\Gamma_s)}\). Then, we have to solve: Find \(P_h \lambda \in H^{1/2}_{s,+}\) such that
\[
\langle WP_h \lambda, \lambda_h - P_h \lambda \rangle \geq \langle W \lambda, \lambda_h - P_h \lambda \rangle \quad \forall \lambda_h \in H^{1/2}_{s,+},
\]
and find \(\Phi(q) \in H^{1/2}_{s,h}\) such that
\[
\langle W \Phi(q), \lambda_h \rangle = d(q, \lambda_h) \quad \forall \lambda_h \in H^{1/2}_{s,h}.
\]
Both systems are small compared with the total size of the problem, because they are only defined on the Signorini part \(\Gamma_s\) of the interface \(\Gamma\). Applying (5.153) to (5.152) we obtain for \(\lambda = \lambda^{(n)} + \phi \Phi(q^{(n)})\),
\[
\langle W \lambda, \lambda_h - P_h \lambda \rangle = \langle W \lambda^{(n)}, \lambda_h - P_h \lambda \rangle + \phi d(q^{(n)}, \lambda_h - P_h \lambda),
\]
and hence, the explicit solution of (5.153) is avoided.
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5.4 Numerical examples

Example 5.1 In the following we present a numerical example for the interface problem with Signorini interface conditions, which is the dual-mixed version of Example 4.3. The geometry is the L-shaped domain shown in Figure 4.3. We take \( \kappa = I_{2 \times 2} \) and vanishing body forces, i.e., \( f = 0 \), Signorini conditions on both large edges of the L-shape and we choose \( u_0 = r^{2/3} \sin \frac{2}{3} (\varphi - \frac{\pi}{2}) \) and \( t_0 = \frac{\partial}{\partial n} u_0 \). Then there holds \( \Delta u_0 = 0 \), and hence

\[
0 = (\Delta u_0, 1)_{L^2(\Omega)} = - (\nabla u_0, \nabla 1)_{(L^2(\Omega))^n} + (1, \frac{\partial}{\partial n} u_0),
\]

that is \( \langle 1, t_0 \rangle = 0 \), whence the corresponding uniqueness requirement (5.6) is fulfilled.

Here we are using rectangles and it was necessary to use \( h = 2h \) for \( H^{1/2}_h \) to fulfill the requirement \( h \leq C_0 h \) of the discrete Babuška-Brezzi condition in Lemma 5.15.

Figure 5.1 and Figure 5.2 show the solution for \( \dim L_h = 3072 \), \( \dim H_h = 6272 \) and \( \dim H^{1/2}_h = 63 \) elements, using rectangular elements. Figure 5.1 shows the solution \( u_h \) (color plot and surface plot) and \( \lambda_h \). This corresponds to Figure 4.5 of the primal formulation. The pictures, i.e., the solutions, are very similar, whether using piecewise constant splines for \( u_h \) in the dual-mixed formulation or piecewise linear and continuous splines for \( u_h \) in the primal formulation. Figure 5.2 shows the absolute value of \( q_h \), i.e., the absolute value of the gradient of \( u \). We can clearly recognize the singularities in the inner corner and the corners which coincide with the change of boundary conditions from Signorini to transmission conditions.

The numerical solutions of the discrete problem \( (M_h) \) are computed with the modification (5.152) of the Uzawa algorithm 5.1. Then, Table 5.1 gives the numbers of outer iterations and the computing time for the Uzawa algorithm with \( \rho = 1.3 \). We notice that the numbers of outer iterations are nearly independent of \( \dim H^{1/2}_{s,h} \). Note also, that \( \dim H^{1/2}_{s,h} \) is very small compared with the total size of the problem \( (M_h) \). The inner linear systems are solved with the GMRES algorithm.

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Table 5.1: Iteration numbers and cpu-times for the Uzawa algorithm
Figure 5.1: Coupling problem, dual formulation \((u, \lambda)\)
Figure 5.2: Coupling problem, dual formulation ($p$)
6.1 Graded Meshes in 2d

6.1.1 Introduction

In this section we present the additive Schwarz method for the hypersingular integral equation of the first kind for the \( h \) version of the Galerkin method on an algebraically graded mesh.

We consider the hypersingular integral equation

\[
Dv(x) := \frac{-1}{\pi} f.p. \int_{\Gamma} \frac{v(y)}{|x-y|^2} \, ds_y = f(x), \quad x \in \Gamma = (-1, 1),
\]

where f.p. denotes a finite part integral in the sense of Hadamard. Let \( \tilde{\Gamma} \) be an arbitrary closed curve containing \( \Gamma \). We define, as in \cite{66}, the Sobolev spaces

\[
\tilde{H}^{1/2}(\Gamma) = \left\{ v|_{\tilde{\Gamma}} : v \in H^{1}_{loc}(\mathbb{R}^2) \text{ and } v|_{\Gamma \setminus \Gamma} = 0 \right\},
\]

and \( H^{-1/2}(\Gamma) \) being the dual space of \( \tilde{H}^{1/2}(\Gamma) \) with respect to the \( L^2 \) inner product on \( \Gamma \). As was shown in \cite{27}, \( D \) is continuous and invertible from \( \tilde{H}^{1/2}(\Gamma) \) to \( H^{-1/2}(\Gamma) \). Moreover, \( D \) is strongly elliptic, i.e., there exists a constant \( \gamma > 0 \) such that

\[
\langle Dv, v \rangle \geq \gamma \| v \|^2_{\tilde{H}^{1/2}(\Gamma)},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \) duality on \( \Gamma \). Hence, \( D \) defines a continuous, positive-definite and symmetric bilinear form \( a(v, w) = \langle Dv, w \rangle \) for \( v, w \in \tilde{H}^{1/2}(\Gamma) \).

It is well known that the solution of (6.1) has unbounded derivatives at the endpoints of the interval \( \Gamma \) \cite{92}. This singular behavior of the solution leads to low convergence rate of order \( h^{1/2} \) for the boundary element Galerkin solution with piecewise linear, continuous trial functions on a quasi-uniform mesh. One can regain the optimal convergence rate of order \( h^{3/2} \) if one uses a mesh, which is algebraically graded towards the endpoints of the interval \( \Gamma \).

Next, let us look at a general setting of Schwarz methods. Let \( V \) be a finite dimensional space, \( a(\cdot, \cdot) \) be a bilinear form, and \( f \) be a linear form on \( V \). Let us consider the abstract variational problem:

Given a finite dimensional space \( V \) find \( u^* \in V \) such that

\[
a(u^*, v) = f(v) \quad \forall v \in V
\]
Chapter 6. Graded and HP-Meshes

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Assume that $V$ is decomposed as

$$V = V_1 + \ldots + V_L,$$

where together with each subspace $V_j$ we associate a projection $P_j : V \to V_j$ defined as

$$a(P_jv, w) = a(v, w) \quad \forall v, w \in V.$$

With these operators we have the additive Schwarz operator

$$P_{ASM} = \sum_{j=1}^L P_j.$$

This Schwarz operator gives a preconditioner for the CG method but can also be used as iterative solver for (6.2). Here we demonstrate the performance for the Galerkin bem applied to the first kind integral equation (6.1) on graded meshes. For theoretical estimates of the condition numbers of the preconditioned equations on quasi-uniform meshes see [72, 94, 74, 90], a corresponding analysis for graded meshes is currently under investigation [96, 73].

Other domain decomposition methods in connection with integral equations are studied, e.g., in [61, 88].

### 6.1.2 Subspace decompositions

First, we consider the algebraically graded mesh. We will define the test and trial space on $(-1, 0)$. The space on the whole interval $\Gamma$ can then be defined by symmetry. On the interval $(-1, 0)$ we use the partition

$$-1 = x_0^l < x_1^l < \ldots < x_{N_l}^l = 0,$$

where $x_i^l = -1 + \left(\frac{i}{N_l}\right)^\beta$, $i = 1, \ldots, N_l$, $l = 1, \ldots, L$, for a constant $\beta \geq 1$. The mesh grading parameter $\beta$ steers the algebraic grading towards the endpoint $-1$.

Let $\{\phi_1^l, \ldots, \phi_{N_l}^l\}$ be the nodal basis for $V^l$, where $N_l = 2^l - 1$ is the dimension of $V^l$.

The subspaces $V^l$ are nested, that is $V^1 \subset \ldots \subset V^l \subset \ldots \subset V^L$.

We then decompose $V^L$ as

$$V^L = \sum_{i=1}^{N_l} V_i^l,$$

with $V_i^l = \text{span}\{\phi_i^l\}$ for $i = 1, \ldots, N_l$. Eventually, $V^L$ is decomposed as

$$V^L = \sum_{l=1}^L \sum_{i=1}^{N_l} V_i^l.$$

The multilevel additive Schwarz operator is now defined as

$$P_{ASM} = \sum_{l=1}^L \sum_{i=1}^{N_l} P_{V_i^l},$$

where $P_{V_i^l} : V^L \to V_i^l$ is defined for any $v \in V^L$ by

$$a(P_{V_i^l}v, w) = a(v, w) \text{ for all } w \in V_i^l.$$
This multilevel method was originally designed in [32] for finite element discretizations of elliptic differential equations. An alternative discussion was given in [16] with the so-called BPX preconditioner. We shall prove in this section that this method is also a good preconditioner for the hypersingular integral equation on graded meshes.

Our convergence analysis is based on two approximation assumptions (A.1) and (A.2) and connects the abstract analysis in [76] (see also Hackbusch [49]) with the approach given by Yserentant [101] for finite element approximations of differential equations on graded meshes. The assumptions themselves have been proved in [96] for graded meshes.

The discrete norm used in [96] is defined by

$$
\forall v \in V^l \text{ with } v = \sum_{i=1}^{N_l} \tilde{v}_l^i \phi_l^i : \quad \|v\|_{k,l}^2 := \sum_{i=1}^{N_l} (\tilde{v}_l^i)^2
$$

The assumptions are

(A.1) \[ \exists K_1 > 0 \forall v \in V^l : \quad \|v\|_a \leq K_1 \|v\|_{b,l} \] (6.5)

(A.2) For \( v \in V^l \) let \( P_{V^{l-1}} v \in V^{l-1} \) be the orthogonal projection w.r.t. \( a(\cdot, \cdot) \) of \( v \) on \( V^{l-1} \) (Galerkin projection). Then

\[ \exists K > 0 \forall v \in V^l : \quad \|(I - P_{V^{l-1}})v\|_{b,l} \leq K \|(I - P_{V^{l-1}})v\|_a. \] (6.6)

We will first prove a bound for the minimum eigenvalue of \( P_{ASM} \).

**Lemma 6.1** There exists a positive constant \( C_0 \) depending on the grading parameter \( \beta \) but independent of the number of levels and the number of mesh points such that

$$
\lambda_{\text{min}}(P_{ASM}) \geq C_0.
$$

**Proof.** We will prove that

$$
\sum_{l=1}^{L} \sum_{i=1}^{N_l} a(v_l^i, v_l^i) \leq C_0^{-1} a(v, v), \quad (6.7)
$$

for some decomposition \( v = \sum_{l=1}^{L} \sum_{i=1}^{N_l} v_l^i \). We define for each \( l = 1, \ldots, L \) a projection \( \tilde{P}_{V^l} : \widetilde{H}^{1/2}(\Gamma) \to V^l \) for any \( v \in \widetilde{H}^{1/2}(\Gamma) \) by

$$
a(\tilde{P}_{V^l} v, \phi) = a(v, \phi) \quad \text{for all } \phi \in V^l,
$$

i.e., \( \tilde{P}_{V^l} \) is the Galerkin projection on \( V^l \). With the introduction of \( \tilde{P}_{V^l} \), we can first decompose \( v \in V^L \) as \( v = \sum_{l=1}^{L} v_l \) by letting

$$
v_l^i := (\tilde{P}_{V^l} - \tilde{P}_{V^{l-1}}) v \quad \text{for any } l = 1, \ldots, L, \quad \text{with } \tilde{P}_{V^0} \equiv 0.
$$

We further decompose \( v_l^i \in V^l \) as

$$
v_l^i = \sum_{i=1}^{N_l} v_l^i, \quad \text{with } v_l^i = \tilde{v}_l^i \phi_l^i \in V_l^l.
$$

We note that from the definition of \( v_l^i \) there holds

$$
a(v_l^i, v_k^i) = 0 \quad \text{for } 1 \leq l \neq k \leq L.
$$
Using Assumptions A.1 and A.2 of [96] we obtain
\[
\sum_{i=1}^{N_l} a(v_i^l, v_i^l) \leq c\|v_i^l\|_{\mathcal{H}_d}^2 = c\|(I - \tilde{P}_{V_{l-1}}) v_i^l\|_{\mathcal{H}_d}^2 \leq cK\|(I - \tilde{P}_{V_{l-1}}) v_i^l\|_{\mathcal{A}}^2 = cK a(v^l, v^l)
\]
due to \(\tilde{P}_{V_{l-1}} = P_{V_{l-1}}\) on \(V^l\).

Using the orthogonality we finally obtain
\[
\sum_{l=1}^{L} \sum_{i=1}^{N_l} a(v_i^l, v_i^l) \leq cK \sum_{l=1}^{L} a(v^l, v^l) = cK a(v, v) =: C_0^{-1} a(v, v).
\]

We will next prove a bound for \(\lambda_{\max}(P_{ASM})\) which depends only logarithmically on the number of mesh points. Let \(T_l = \sum_{i=1}^{N_l} P_{V_i^l}\). Then, we can write \(P_{ASM} = \sum_{l=1}^{L} T_l\). The following Cauchy-Schwarz inequality is essential for obtaining a bound for \(\lambda_{\max}(P_{ASM})\).

**Lemma 6.2** There exists a constant \(c > 0\) such that for any \(v \in V^k\) where \(1 \leq k \leq l \leq L\) there holds
\[
a(T_l v, v) \leq c a(v, v)
\]

**Proof.** Since \(V_i^l = \text{span}\{\phi_i^l\}\) we have for any \(v \in V^L\)
\[
P_{V_i^l} v = \frac{a(v, \phi_i^l)}{a(\phi_i^l, \phi_i^l)} \phi_i^l,
\]
and therefore
\[
T_l v = \sum_{i=1}^{N_l} \frac{a(v, \phi_i^l)}{a(\phi_i^l, \phi_i^l)} \phi_i^l.
\]
Hence
\[
a(T_l v, v) = \sum_{i=1}^{N_l} \frac{a(v, \phi_i^l)^2}{a(\phi_i^l, \phi_i^l)}.
\]

Let \(M_l^l = (M_{ij}^l)\) with \(M_{ij}^l = a(\phi_i^l, \phi_j^l)\) be the Galerkin matrix of level \(l\). Let \(C^{kl} \in \mathbb{R}^{N_k \times N_l}\) be defined by
\[
\phi_i^k = \sum_{j=1}^{N_l} c_{ij}^{kl} \phi_j^l, \quad k \leq l.
\]

Using Lemma 6.4 and \(v = \sum_{i=1}^{N_k} v_i \phi_i^k \in V_k, \ \vec{v} = (v_1, \ldots, v_{N_k})^T\) we can estimate
\[
a(T_l v, v) = \sum_{i=1}^{N_l} \frac{a(v, \phi_i^l)^2}{a(\phi_i^l, \phi_i^l)} \leq c \sum_{i=1}^{N_l} a(v, \phi_i^l)^2 = c \sum_{i=1}^{N_l} (v^T C^{kl} M_{ij}^l)^2 = c \bar{v}^T C^{kl} M^l M^l (C^{kl})^T \bar{v}
\]
where \(c\) depends only on the grading parameter \(\beta\).

\(M^l\) is a symmetric and positive definite matrix. Therefore we can decompose
\[
M^l M^l = (M^l)^{1/2} M^l (M^l)^{1/2}
\]
and obtain
\[ a(T_l v, v) \leq c \bar{v}^T C^{kl} M_l^l M_l^l (C^{kl})^T \bar{v} \]
\[ = c \bar{v}^T C^{kl} (M_l^l)^{1/2} M_l^l (M_l^l)^{1/2} (C^{kl})^T \bar{v} \]
\[ \leq c \lambda_{\max}(M_l^l) \bar{v}^T C^{kl} (M_l^l)^{1/2} (M_l^l)^{1/2} (C^{kl})^T \bar{v} \]
\[ = c \lambda_{\max}(M_l^l) \bar{v}^T C^{kl} M_l^l (C^{kl})^T \bar{v} \]
\[ = c \lambda_{\max}(M_l^l) \bar{v}^T M_l^k \bar{v} \]
\[ = c \lambda_{\max}(M_l^l) a(v, v). \]

Let \( u = \sum_{i=1}^{N_l} u_i \phi_i^l \). With Assumption A.1 from [96] we have
\[ \bar{u}^T M_l^l \bar{u} = a(u, u) \leq K_1^2 \sum_{i=1}^{N_l} u_i^2. \]

Therefore \( \lambda_{\max}(M_l^l) \) is bounded independent of the number of mesh nodes. Altogether we obtain
\[ a(T_l v, v) \leq c \lambda_{\max}(M_l^l) a(v, v) \leq c K_1^2 a(v, v). \]

Lemma 6.3 There exists a positive constant \( C_1 \) depending on the grading parameter \( \beta \) but being independent of the number of levels \( L \) and the number of mesh points such that
\[ \lambda_{\max}(P_{ASM}) \leq L^2 C_1. \]

Proof. The lemma is proved if one proves
\[ a(P_{ASM} v, v) \leq L^2 C_1 a(v, v) \quad \text{for any } v \in V^L. \quad (6.8) \]
The proof is similar to [13, Theorem 3.1] and is included here for completeness. For \( l = 1, \ldots, L \) let \( P_l : V^L \to V^l \) be defined for any \( v \in V^L \) as
\[ a(P_l v, w) = a(v, w) \quad \text{for any } w \in V^l. \]

Then for any \( v \in V^L \) we can write
\[ P_l v = \sum_{k=1}^{l} (P_k - P_{k-1}) v, \]
where \( P_0 = 0 \). Hence
\[ a(P_{ASM} v, v) = \sum_{l=1}^{L} a(T_l v, v) = \sum_{l=1}^{L} a(T_l v, P_l v) = \sum_{l=1}^{L} \sum_{k=1}^{l} a(T_l v, (P_k - P_{k-1}) v). \]

Since \( a(T_l \cdot, \cdot) \) is symmetric and positive definite we can use Cauchy-Schwarz’s inequality...
with respect to that quadratic form and then use Lemma 6.2 to obtain, for any \( \eta > 0 \),

\[
a(\text{ASM} v, v) \leq \sum_{l=1}^{L} \sum_{k=1}^{l} a(T_l v, v) \frac{1}{L} \sum_{l=1}^{L} \sum_{k=1}^{l} a((P_k - P_{k-1}) v, (P_k - P_{k-1}) v)^{1/2}
\]

\[
\leq c \left( \frac{\eta}{L} \sum_{l=1}^{L} \sum_{k=1}^{l} a(T_l v, v) + \frac{L}{4\eta} \sum_{l=1}^{L} \sum_{k=1}^{l} a((P_k - P_{k-1}) v, v) \right)
\]

\[
\leq c \left( \eta \sum_{l=1}^{L} a(T_l v, v) + \frac{L^2}{4\eta} \sum_{k=1}^{L} a((P_k - P_{k-1}) v, v) \right)
\]

\[
= c \left( \eta a(\text{ASM} v, v) + \frac{L^2}{4\eta} a(v, v) \right).
\]

By choosing \( \eta \) sufficiently small we obtain (6.8); therefore the lemma is proved. \( \Box \)

**Lemma 6.4** Let \( \phi(x) \) be a hat-function with \( \phi(-h_1) = 0 \), \( \phi(0) = 1 \) and \( \phi(h_2) = 0 \).

Then we have

\[
a(\phi, \phi) = \frac{1}{\pi} \left( \left( 1 + \frac{h_1}{h_2} \right) \log \left( 1 + \frac{h_2}{h_1} \right) + \left( 1 + \frac{h_2}{h_1} \right) \log \left( 1 + \frac{h_1}{h_2} \right) \right)
\]

and the bounds

\[
\frac{1}{\pi} 2 \log 2 \leq a(\phi, \phi) \leq \frac{1}{\pi} \left( 2 + \frac{h_1}{h_2} + \frac{h_2}{h_1} \right).
\]

### 6.1.3 Numerical results

We consider the equation (6.1) with the right hand side \( f(x) \equiv 1 \). We solve the Galerkin equations (6.2) for the algebraically graded mesh using the additive Schwarz method. We use the additive Schwarz operator as a preconditioner of the conjugate gradient (CG) method.

Tab. 6.1 and Fig. 6.1 give the condition numbers of the unpreconditioned Galerkin matrix \( A_N \) with respect to different numbers of degrees of freedom \( N \) and grading parameters \( \beta \). We note that the condition numbers grow linearly with \( N \). Tab. 6.2 and Fig. 6.2 give the corresponding condition numbers of \( P_{\text{ASM}} \), which grow only logarithmically.

<table>
<thead>
<tr>
<th>( \beta ) ( N )</th>
<th>7</th>
<th>15</th>
<th>31</th>
<th>63</th>
<th>127</th>
<th>255</th>
<th>511</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>3.8765</td>
<td>7.7629</td>
<td>15.5817</td>
<td>31.2459</td>
<td>62.5873</td>
<td>125.2775</td>
<td>250.4733</td>
</tr>
<tr>
<td>2.0</td>
<td>2.7469</td>
<td>5.4310</td>
<td>10.8338</td>
<td>21.6572</td>
<td>43.3089</td>
<td>86.6153</td>
<td>173.1770</td>
</tr>
<tr>
<td>3.0</td>
<td>2.3367</td>
<td>4.4641</td>
<td>8.8173</td>
<td>17.5753</td>
<td>35.0910</td>
<td>70.2228</td>
<td>142.4616</td>
</tr>
<tr>
<td>4.0</td>
<td>2.2293</td>
<td>4.1911</td>
<td>8.2400</td>
<td>16.4050</td>
<td>32.2620</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6.1: Hypersingular integral equation, algebraically graded \( h \) version without preconditioner, condition numbers \( \kappa(A_N) \)


\begin{center}
\begin{tabular}{c|cccccccc}
$\beta \backslash N$ & 7 & 15 & 31 & 63 & 127 & 255 & 511 \\
\hline
3.0 & 2.9788 & 3.7577 & 4.3090 & 4.6811 & 4.9294 & 5.0949 & 5.3210 \\
4.0 & 3.0237 & 3.8515 & 4.4544 & 4.8629 & 5.2267 & & \\
\end{tabular}
\end{center}

Table 6.2: Hypersingular integral equation, algebraically graded $h$ version, condition numbers $\kappa(P_{ASM})$ for the additive Schwarz method

---

Figure 6.1: Hypersingular integral equation, algebraically graded $h$ version without preconditioner, condition numbers $\kappa(A_N)$
Figure 6.2: Hypersingular integral equation, algebraically graded $h$ version, condition numbers $\kappa(P_{ASM})$ for the additive Schwarz method
6.2 Multiplicative Schwarz methods for \textit{hp} meshes in 3d

We study the multiplicative Schwarz method for the \textit{p} version boundary element method for a weakly singular integral equation of the first kind in 3D. We prove that the rate of convergence of the multiplicative Schwarz operator for the \textit{p} version grows only logarithmically in \(p\) and is independent of \(h\) for the 2-level method.

6.2.1 General setting of multiplicative Schwarz methods

Multiplicative (and additive) Schwarz methods are in general defined via a subspace decomposition of the space of test and trial functions together with projections onto these subspaces. More precisely, let
\[ V = V_0 + V_1 + \cdots + V_{N-1}, \tag{6.9} \]
and let \(P_j : V \to V_j, \ j = 0, \ldots, N - 1\), be projections defined by
\[ a(P_j v, \phi) = a(v, \phi) \quad \forall v \in V, \ \phi \in V_j. \]
Here \(a(\cdot, \cdot)\) is a symmetric and positive-definite bilinear form on \(V\). The multiplicative Schwarz operator is then defined as
\[ P_{MS} = I - E_{N-1}, \]
where \(I\) is the identity map and
\[ E_{N-1} = (I - P_0)(I - P_1) \cdots (I - P_{N-1}) \]
is the error propagation operator. We note that
\[ P_{AS} = \sum_{j=0}^{N-1} P_j \]
is the corresponding additive Schwarz operator.

The analysis in [15] and [100] suggest that we would have to prove a strengthened Cauchy-Schwarz inequality corresponding to the decomposition (6.9). The following theorem [74, Theorem 2.2] shows that it is possible to avoid the coarse grid subspace \(V_0\) in this inequality and to use instead bounds for the maximum and minimum eigenvalues of the additive Schwarz operator.

\textbf{Theorem 6.1} [74, Theorem 2.2] Let \(\Theta\) be a \((N - 1) \times (N - 1)\) matrix with elements \(\theta_{ij}, \ i, j = 1, \ldots, N - 1\) defined by
\[ \theta_{ij} := \sup_{u \in V_i, v \in V_j} \frac{a(u, v)}{a(u, u)^{1/2} a(v, v)^{1/2}}. \]
If there exist positive constants \(C_0, C_2\) satisfying
\[ C_0 a(v, v) \leq a(P_{AS} v, v) \leq C_2 a(v, v) \quad \forall v \in V, \]
and a constant \(C_1\) satisfying
\[ C_1 \geq 2 \max\{C_2, \|\Theta\|_2^2\}, \]
then there holds
\[ \|E_{N-1} v\|_a^2 \leq \left(1 - \frac{C_0}{C_1}\right) \|v\|_a^2. \]
Here \(\|\cdot\|_a\) is the norm given by the bilinear form \(a(\cdot, \cdot)\).
6.2.2 Weakly singular integral equation

We consider the weakly singular integral equation

\[ V \psi(x) := -\frac{1}{2\pi} \int_{\Gamma} \frac{\psi(y)}{|x-y|} \, ds_y = f(x), \quad x \in \Gamma = (-1, 1)^2. \tag{6.10} \]

As it was shown in [27], \( V \) is continuous and invertible from \( \tilde{H}^{-1/2}(\Gamma) \) to \( H^{1/2}(\Gamma) \). Moreover, \( V \) is strongly elliptic, i.e., there exists a constant \( \gamma > 0 \) such that

\[ \langle V \psi, \psi \rangle \geq \gamma \| \psi \|_{\tilde{H}^{-1/2}(\Gamma)}^2, \]

where \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \) duality on \( \Gamma \). Hence \( V \) defines a continuous, positive-definite and symmetric bilinear form \( a(v, w) = \langle Vv, w \rangle \) for \( v, w \in H^{-1/2}(\Gamma) \).

The quasi-uniform \( hp \) version for the weakly singular operator

We consider a uniform mesh of size \( h \) on \( \Gamma \) as follows

\[ x_j = -1 + jh, \quad h = \frac{2}{n}, \quad j = 0, \ldots, n, \tag{6.11} \]

\[ \Gamma = \bigcup_{i,j=1}^n \Gamma_{ij}, \quad \Gamma_{ij} := (x_{i-1}, x_i) \times (x_{j-1}, x_j). \]

We define on the mesh (6.11) the space \( V^p \) of piecewise continuous functions on \( \Gamma \) whose restrictions on \( \Gamma_{ij} := (x_{i-1}, x_i) \times (x_{j-1}, x_j) \), \( i, j = 1, \ldots, n \), are polynomials of degree at most \( p \) in \( x \)- and \( y \)-direction, \( p \geq 1 \). For the \( p \) version of the Galerkin scheme, we approximate the solution of (6.10) by functions in \( V^p \) and increase the accuracy of the approximation not by reducing \( h \) (which is fixed) but by increasing \( p \). More explicitly, the \( p \) version boundary element method for equation (6.10) reads as:

Find \( u_p^* \in V^p \) such that

\[ a(u_p^*, v_p) = \langle f, v_p \rangle \quad \text{for any} \quad v_p \in V^p. \tag{6.12} \]

The stability and convergence of the scheme (6.12) was proved in [91]. Note that the dimension of \( V^p \) is \( \dim V^p = p^2 n^2 \). Choosing a basis for \( V^p \), we derive from (6.12) a system of equations to be solved for \( u_p^* \). In practice, we use the following basis.

For \( q = 0, \ldots, p \) and \( j = 1, \ldots, n \) we define \( L_{q,j} \) as the affine image onto \( \Gamma_{ij} \) of the Legendre polynomial of degree \( q \). Then we define

\[ L_{i,j}^{(k,l)}(x,y) = L_{k,i}(x)L_{l,j}(y) \quad \text{on} \quad \Gamma_{ij}. \]

We extend \( L_{i,j}^{(k,l)} \) by 0 outside \( \Gamma_{i,j} \). It is clear that

\[ \mathcal{B} = \{ L_{1,1}^{(0,0)}, \ldots, L_{n,n}^{(0,0)}, \ldots, L_{1,1}^{(p,p)}, \ldots, L_{n,n}^{(p,p)} \} \]

is a basis for \( V^p \). Solving the equation (6.12) amounts to solving

\[ A_N \psi_N = f_N, \tag{6.13} \]

where the matrix \( A_N \) has entries \( a(v, w) \) with \( v, w \in \mathcal{B} \). The condition number of (6.13) grows at least like \( p^2 \) and at most like \( p^3 \). We will define a multiplicative Schwarz method.
to solve a preconditioned system which has a condition number growing significantly slower than \( p^2 \) instead of solving (6.13).

We decompose \( V^p \) as a direct sum

\[
V^p = V_0 \bigoplus_{i,j=1}^{n} V_{i,j}^p,
\]

where

\[
V_0 := \text{span}\{L_{i,j}^{(0,k)}, L_{i,j}^{(k,0)} : i, j = 1, \ldots, n; k = 0, \ldots, p\},
\]

and

\[
V_{i,j}^p := \text{span}\{L_{i,j}^{(1,1)}, \ldots, L_{i,j}^{(p,p)}\} \text{ for } i, j = 1, \ldots, n.
\]

The space \( V_0 \) serves as a coarse space. We note that functions in \( V_{i,j}^p \) are supported in \( \Gamma_{i,j}^0 \).

The multiplicative Schwarz operator is defined as above, i.e.,

\[
P_{\text{MS}} = I - E_N,
\]

where

\[
E_N = (I - P_0) \prod_{i,j=1}^{n} (I - P_{i,j}).
\]

We have (see [52])

**Lemma 6.5** There exist constants \( C_0, C_2 \) independent of \( p \) and \( n \) such that for any \( v \in V^p \) there holds

\[
C_0 a(v, v) \leq a(Pv, v) \leq C_2 (1 + \log(p + 1))^2 a(v, v),
\]

where \( P := P_0 + \sum_{i,j=1}^{n} P_{i,j} \).

Using the abstract theorem [74, Theorem 2.2] we can prove

**Theorem 6.2** Let \( C_0, C_2 \) given by Lemma 6.5 and \( C_3 \) for the strengthened Cauchy-Schwarz inequality given by Lemma 6.6. If \( C_1 := 2 \max\{C_2, C_3\} \), then for any \( v \in V^p \) there holds

\[
\|E_N v\|_{\tilde{H}^{1/2}(\Gamma)} \leq \left(1 - \frac{C_0}{C_1(1 + \log^2(p + 1))^2}\right)^{1/2} \|v\|_{\tilde{H}^{1/2}(\Gamma)}.
\]

### 6.2.3 Numerical results

We consider the weakly singular integral equation (6.10) with the right hand side \( f(x) \equiv 1 \). We solve the Galerkin equations (6.12) for the \( p \) version by the multiplicative Schwarz algorithm. The numerical results in Fig. 6.3 and Tab. 6.3 show the expected behavior of the contraction rate. Iteration numbers and CPU-times for the \( p \) version of the weakly singular integral equation are given in Fig. 6.4 and Fig. 6.5. We note that the iteration numbers do not depend on \( n \) and increase only slowly with \( p \). The number of degrees of freedom is given by \( N = n^2 p^2 \).

We calculate the elements of the Galerkin matrix semi-numerically by applying a graded quadrature rule to the analytically computed single layer potential to achieve sufficient precision for polynomial degree \( p = 10 \).
We have to note that our subspace decomposition in this case is actually a reordering of the basis functions. Therefore the projections involved are simplified considerably. The contraction rates are evaluated by computing the error reduction operator $E$ in matrix form explicitly and determining the largest eigenvalue.

For the computation of the iteration numbers and CPU-times we have always used the stopping criterion

$$\frac{\|x^k - x^{k-1}\|_2}{\|x^k\|_2} \leq \varepsilon, \quad \varepsilon = 10^{-10},$$

where $\| \cdot \|_2$ is the Euclidean norm and $x^{k-1}, x^k$ are successive iterates. Stopping criteria which are based on residuals, are more expensive to compute, because the residual of the iterative solution is not a byproduct of the computation (cf. the CG-algorithm). Therefore (6.15) is the cheapest stopping criterion, which is available. The numerical experiments were performed on a Sun-E450 (480MHz) at the University of Hannover using the program system maiprogs [69]. In the case of the 2-level method applied to the $p$ version the subspace decomposition is a direct sum (c.f. [72]), i.e., the multiplicative Schwarz method can be implemented as a block Jacobi method.

<table>
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<th>$n=4$</th>
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</table>

Table 6.3: Weakly singular integral equation, $p$ version
Figure 6.3: Contraction rates for the weakly singular integral equation, $p$ version

Figure 6.4: Iteration numbers for the weakly singular integral equation, $p$ version
Figure 6.5: Computing times for the weakly singular integral equation, $p$ version
6.2.4 Proof of technical lemmas

**Lemma 6.6 (Single layer potential)** Let \(1 \leq i_1, i_2, j_1, j_2 \leq n\). Then there exists a constant \(C\) independent of \(i_1, i_2, j_1, j_2, p\) such that

\[
\theta_{(i_1, i_2), (j_1, j_2)} := \cos(V_{i_1}^p, V_{j_1}^p) = \sup_{u \in V_{i_1}^p, v \in V_{j_1}^p} \frac{a(u, v)}{a(u, u)^{1/2} a(v, v)^{1/2}} \leq \frac{C}{(((i_1 - j_1) + 1)^2 + (|j_2 - j_2| + 1)^2)^{5/2}}
\]

(6.16)

and with \(\Theta = \{\theta_{(i_1, i_2), (j_1, j_2)}\}_{1 \leq i_1, i_2, j_1, j_2 \leq n}\) we have

\[
\|\Theta\|_2 \leq C < \infty.
\]

(6.17)

**Proof.** Let \(u = \sum_{k,l=1}^{p} u_{k,l} L_{k,i_1} L_{l,i_2}\) and \(v = \sum_{m,n=1}^{p} v_{k,l} L_{m,j_1} L_{m,j_2}\). Then we have

\[
a(u, v) = \sum_{k,l=1}^{p} \sum_{m,n=1}^{p} u_{k,l} v_{m,n} \langle V L_{k,i_1} L_{l,i_2}, L_{m,j_1} L_{m,j_2} \rangle.
\]

(6.18)

From Lemma 6.8 we have

\[
\text{dist}(\Gamma_{i_1, i_2}, \Gamma_{j_1, j_2}) = h \sqrt{(|i_1 - j_1| - 1)^2 + (|j_2 - j_2| - 1)^2} = hd_{i_1, i_2, j_1, j_2}
\]

\[
\langle V L_{k,i_1} L_{l,i_2}, L_{m,j_1} L_{m,j_2} \rangle \leq \frac{1}{2\pi} \frac{4h^4}{\sqrt{(2k + 1)(2l + 1)(2m + 1)(2n + 1)}} \frac{(k + l + m + n)^3 (h/2)^{2+k+l+m+n}}{\text{dist}(\Gamma_{i_1, i_2}, \Gamma_{j_1, j_2})^{k+l+m+n+1}}
\]

\[
\leq \frac{1}{2\pi} \frac{4h^3}{\sqrt{(2k + 1)(2l + 1)(2m + 1)(2n + 1)}} \frac{(k + l + m + n)^3 (3/2)^{k+l+m+n}}{d_{i_1, i_2, j_1, j_2}^{k+l+m+n+1}}.
\]

(6.19)

From (6.18) and (6.19) we get

\[
|a(u, v)| \leq C \sum_{k,l=1}^{p} \sum_{m,n=1}^{p} \frac{|u_{k,l}| |v_{m,n}| 4h^3(k + l + m + n)^3 (3/2)^{k+l+m+n}}{\sqrt{(2k + 1)(2l + 1)(2m + 1)(2n + 1)}} \times
\]

\[
\left( \begin{array}{c} k^3/2 |u_{k,l}| \times k^3/2 |v_{m,n}| \\ k \times \frac{4k!}{\prod_{i=1}^{k} i} \end{array} \right)
\]

\[
\leq C \left( \sum_{k,l=1}^{p} \left( \frac{k^3/2 |u_{k,l}|}{k \sqrt{(2k + 1)(2l + 1)}} \right)^{1/2} \left( \sum_{m,n=1}^{p} \left( \frac{h^3/2 |v_{m,n}|}{m \sqrt{(2m + 1)(2n + 1)}} \right)^{1/2} \right)^{1/2} \times
\]

\[
\left\{ \left( \frac{4k!}{\prod_{i=1}^{k} i} \right)^{m,n=1,...,p} \right\}_{k,l=1,...,p} \right)_2
\]

(6.20)
For $3/(2d_{i_1,i_2,j_1,j_2}) < 1$ we have

$$
\left\| \frac{4kl mn (k + l + m + n)^3(3/2)^{k+l+m+n}}{d_{i_1,i_2,j_1,j_2}^{k+l+m+n+1}} \right\|_{k,l=1,\ldots,p}^{m,n=1,\ldots,p} \leq \frac{4kl mn (k + l + m + n)^3(3/2)^{k+l+m+n+1}}{d_{i_1,i_2,j_1,j_2}^{k+l+m+n+1}} \leq C \sum_{m,n=1}^{p} \frac{mn (m + n)^3(3/2)^{2+m+n}}{d_{i_1,i_2,j_1,j_2}^{m+n+3}} \leq C \frac{(3/2)^4}{d_{i_1,i_2,j_1,j_2}^{p}} \tag{6.21}
$$

Due to Lemma 6.9 we have (note the scaling)

$$
\left( \sum_{k,l=1}^{p} \left( \frac{h^{3/2} |u_{k,l}|}{k l \sqrt{(2k + 1)(2l + 1)}} \right)^2 \right)^{1/2} \leq C \|u\|_{H^{-1/2}(V_{1,i_1,j_2})} \leq C \langle Vu, u \rangle^{1/2} = Ca(u, u)^{1/2}.
$$

Using (6.20), (6.21) and (6.22) we get for $\max(|i_1 - j_1|, |i_2 - j_2|) \geq 3$

$$
\theta_{(i_1,i_2),(j_1,j_2)} := \cos(V_{1,i_2}^p, V_{j_1,j_2}^p) = \sup_{u \in V_{1,i_1,j_2}^p, v \in V_{j_1,j_2}^p} \frac{a(u, v)}{a(u, u)^{1/2}a(v, v)^{1/2}} \leq \frac{C}{d_{i_1,i_2,j_1,j_2}^p} \tag{6.23}
$$

Due to the Cauchy-Schwarz inequality we have $\theta_{(i_1,i_2),(j_1,j_2)} \leq 1$ for $\max(|i_1 - j_1|, |i_2 - j_2|) \leq 2$ independent of $p$ and $n$. Therefore we have proved (6.16)

$$
\theta_{(i_1,i_2),(j_1,j_2)} := \cos(V_{1,i_2}^p, V_{j_1,j_2}^p) = \sup_{u \in V_{1,i_1,j_2}^p, v \in V_{j_1,j_2}^p} \frac{a(u, v)}{a(u, u)^{1/2}a(v, v)^{1/2}} \leq \frac{C}{(\langle |i_1 - j_1| + 1 \rangle^2 + \langle |i_2 - j_2| + 1 \rangle^2)^{5/2}}.
$$

Finally we have

$$
\|\Theta\|_2 \leq \|\Theta\|_\infty = \max_{1 \leq i_1, i_2 \leq n} \sum_{j_1,j_2=1}^{n} |\theta_{(i_1,i_2),(j_1,j_2)}| \leq C \max_{1 \leq i_1, i_2 \leq n} \sum_{j_1,j_2=1}^{n} \frac{1}{(\langle |i_1 - j_1| + 1 \rangle^2 + \langle |i_2 - j_2| + 1 \rangle^2)^{5/2}} \leq C \sum_{j_1,j_2=1}^{n} \frac{1}{(j_1^2 + j_2^2)^{5/2}} < \infty. \tag{6.24}
$$

\[\square\]

Lemma 6.7 There holds

$$
\left| \partial_{x_j}^k \partial_{y^l}^{j^l} r^{-1} \right| \leq 3^{k+l} (k + l)! r^{-(k+l+1)}. \tag{6.25}
$$
**Proof.** Let \( f^{(k,l)}(\theta) \) a trigonometric polynomial of degree \( k + l \). Then we have

\[
\partial_x^k \partial_y^l f^{(k,l)}(\theta) = f^{(k,l)}(\theta) r^{(k+l+1)}. \tag{6.26}
\]

By applying \( \partial_x = \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta \) and \( \partial_y = \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta \) we obtain two recurrence formulae for the trigonometric polynomials

\[
f^{(k+1,l)}(\theta) = -(k + l + 1) \cos \theta f^{(k,l)}(\theta) - \sin \theta \partial_\theta f^{(k,l)}(\theta), \tag{6.27}
\]
\[
f^{(k,l+1)}(\theta) = -(k + l + 1) \sin \theta f^{(k,l)}(\theta) + \cos \theta \partial_\theta f^{(k,l)}(\theta). \tag{6.28}
\]

Let \( f^{(k,l)}(\theta) = \sum_{n=0}^{k+l} J_n^{(k,l)} \sin^n \theta \cos^{k-l} \theta \). Using

\[
\partial_\theta f^{(k,l)}(\theta) = \sum_{n=0}^{k+l} j_n^{(k,l)} \left( n \sin^{n-1} \theta \cos^{k+l+1-n} \theta - (k + l - n) \sin^{n+1} \theta \cos^{k+l-n-1} \theta \right)
\]

we have also two recurrence formulae for the coefficients of the trigonometric polynomials

\[
j_n^{(k+1,l)} = -(k + l + 1) j_n^{(k,l)} - n j_{n-1}^{(k,l)} + (k + l + 2 - n) j_{n-2}^{(k,l)},
\]
\[
j_n^{(k,l+1)} = -(k + l + 1) j_n^{(k,l)} - (k + l + 1 - n) j_{n-1}^{(k,l)} + n j_{n+1}^{(k,l)}.
\]

Let

\[
F^{(k,l)} = \sum_{n=0}^{k+l} |j_n^{(k,l)}|.
\]

A close inspection of the recurrence formulae above leads to

\[
F^{(k+1,l)} \leq 3(k + l + 1) F^{(k,l)}, \tag{6.30}
\]
\[
F^{(k,l+1)} \leq 3(k + l + 1) F^{(k,l)}. \tag{6.31}
\]

From \( F^{(0,0)} = 1 \) we obtain by induction \( F^{(k,l)} \leq 3^{k+l}(k + l)! \). Then we have

\[
|\partial_x^k \partial_y^l f^{(k,l)}| \leq F^{(k,l)} r^{-(k+l+1)} \leq 3^{k+l}(k + l)!.
\]

**Lemma 6.8** Let \( L_{p_{1,x},Q_1}(x_1) L_{p_{1,y},Q_1}(x_2) \) be the product of two Legendre polynomials of degree \( p_{1,x} \) resp. \( p_{1,y} \) linearly transformed onto the open rectangle \( Q_1 \subset \mathbb{R}^2 \) with side lengths \( h_{1,x}, h_{1,y} \). \( L_{p_{2,x},Q_1}(x_1) L_{p_{2,y},Q_2}(x_2) \) is supposed to be extended by 0 outside \( Q_1 \) where necessary. Let \( Q_2 \subset \mathbb{R}^2 \) be another open rectangle with \( Q_1 \cap Q_2 = \emptyset \). Let \( h = \max(h_{1,x}, h_{1,y}, h_{2,x}, h_{2,y}) \) the maximal side length of \( Q_1 \) and \( Q_2 \). Then there holds

\[
|\langle V L_{p_{1,x},Q_1} L_{p_{1,y},Q_1}, L_{p_{2,y},Q_2} L_{p_{2,x},Q_2} \rangle| \leq \frac{1}{2\pi} \frac{1}{\sqrt{(2p_{1,x} + 1)(2p_{1,y} + 1)(2p_{2,y} + 1)(2p_{2,x} + 1)}} \frac{|p|^3 (3/2)|p|}{\text{dist}(Q_1, Q_2)^{p_{1,y}^{p_{2,y}}} h^{p_{2,y}}} \tag{6.33}
\]

**Proof.** Let \( x_0 = (x_{1,0}, x_{2,0}) \) be the midpoint of \( Q_1 \) and \( y_0 = (y_{1,0}, y_{2,0}) \) be the midpoint of \( Q_2 \). Let \( |p| = p_{1,x} + p_{1,y} + p_{2,y} + p_{2,x} \). \( D^a_x = \partial_x^{a_1} \partial_x^{a_2} \) and \( (x - x_0)^a = (x - x_{1,0})^{a_1}(x_{2,0} - x_{2,0})^{a_2} \). Then there holds, using the Taylor expansion of \( |x - y|^{-1} = |(x_1, x_2) - (y_1, y_2)|^{-1} \)

\[
|x - y|^{-1} = \sum_{\nu=0}^{|p|-1} \sum_{|\alpha| + |\beta| = \nu} D^a_x D^b_y|x_0 - y_0|^{-1} (x - x_0)^a (y - y_0)^b + R_{|p|}(x, y) \tag{6.34}
\]
and

\[ R_{[p]}(x, y) = \sum_{|\alpha| + |\beta| = |p|} \frac{D_\alpha^2 D_\beta^2 |x_0 + \theta(x - x_0) - y_0 - \theta(y - y_0)|^{-1}}{|p|!} (x - x_0)^\alpha (y - y_0)^\beta \]  

(6.35)

with \( \theta \in (0, 1) \). Due to the orthogonal properties of the Legendre polynomials the first sum vanishes and we have

\[
\langle VL_{p_1, x, Q_1} L_{p_1, y, Q_1}, L_{p_2, Q_2} L_{p_2, Q_2} \rangle = \\
= \frac{1}{2\pi} \int_{Q_1} \int_{Q_2} L_{p_1, x, Q_1} L_{p_1, y, Q_1} L_{p_2, Q_2} L_{p_2, Q_2} |x - y|^{-1} ds_y ds_x \\
= \frac{1}{2\pi} \int_{Q_1} \int_{Q_2} L_{p_1, x, Q_1} L_{p_1, y, Q_1} L_{p_2, Q_2} L_{p_2, Q_2} R_{[p]}(x, y) ds_y ds_x. 
\]  

(6.36)

Applying the Cauchy-Schwarz inequality two times we obtain

\[
|\langle VL_{p_1, x, Q_1} L_{p_1, y, Q_1}, L_{p_2, Q_2} L_{p_2, Q_2} \rangle| \leq \\
\leq \frac{1}{2\pi} \left( \int_{Q_1} L_{p_1, x, Q_1} L_{p_1, y, Q_1} ds_x \right)^{1/2} \left( \int_{Q_2} L_{p_2, Q_2} L_{p_2, Q_2} ds_y \right)^{1/2} \times \\
\times \left( \int_{Q_1} \int_{Q_2} R_{[p]}^2(x, y) ds_y ds_x \right)^{1/2} \\
= \frac{1}{2\pi} \frac{4|Q_1|^{1/2}|Q_2|^{1/2}}{\sqrt{(2p_1, x_1 + 1)(2p_1, y_1 + 1)(2p_2, x_2 + 1)(2p_2, y_2 + 1)}} \left( \int_{Q_1} \int_{Q_2} R_{[p]}^2(x, y) ds_y ds_x \right)^{1/2} \\
\leq \frac{1}{2\pi} \frac{4|Q_1| |Q_2|}{\sqrt{(2p_1, x_1 + 1)(2p_1, y_1 + 1)(2p_2, x_2 + 1)(2p_2, y_2 + 1)}} \sup_{(x, y) \in Q_1 \times Q_2} |R_{[p]}(x, y)|. 
\]  

(6.37)

Due to 6.7 we have

\[
|D_x^\alpha D_y^\beta |x - y|^{-1}| \leq \frac{3^{\alpha + \beta} |\alpha|! |\beta|!}{|x - y|^{\alpha + \beta + 1}}. 
\]  

(6.38)

Therefore we obtain

\[
\sup_{(x, y) \in Q_1 \times Q_2} |R_{[p]}(x, y)| \leq \\
\leq \frac{1}{2\pi} \sum_{(x, y) \in Q_1 \times Q_2} |x_0 + \theta(x - x_0) - y_0 - \theta(y - y_0)|^{p+1} \frac{1}{|p|!} |x - x_0|^{\alpha} |y - y_0|^{\beta} \\
\leq \frac{3^{p+1}}{\text{dist}(Q_1, Q_2)^{p+1}} \sum_{|\alpha| + |\beta| = |p|} |\alpha|! |\beta|! \max(h_{i,1/2}, h_{i,1/2}, h_{j,1/2}, h_{j,1/2}, h_{j,2/2}, h_{j,2/2})^{p+1} \\
\leq \frac{3^{p+1}}{\text{dist}(Q_1, Q_2)^{p+1}} \max(h_{i,1/2}, h_{i,1/2}, h_{j,1/2}, h_{j,1/2}, h_{j,2/2}, h_{j,2/2})^{p+1} \\
= \frac{3^{p+1}}{\text{dist}(Q_1, Q_2)^{p+1}} \max(h_{i,1/2}, h_{i,1/2}, h_{j,1/2}, h_{j,2/2})^{p+1}. 
\]  

(6.39)
Finally, we have

\[
|\langle VL_{p_1,x}Q_1, L_{p_1,y}Q_1, L_{p_2,x}Q_2, L_{p_2,y}Q_2 \rangle| \leq \frac{1}{2\pi} \frac{|Q_1||Q_2|}{(2p_1+1)(2p_2+1)(2p_2, 2p_1 + 1)} \frac{|p|^3 (3/2)^{|p|}}{\max(h_{i,1}, h_{i,2}, h_{j,1}, h_{j,2})^{1/2}}.
\]

\[\square\]

**Lemma 6.9** Let \( Q = (-1, 1)^2 \) and \( u = \sum_{k=1}^{p_1} \sum_{l=1}^{p_2} u_{k,l} L_k L_l. \) Then there exists a constant \( C \) independent of \( u \) and \( p_1, p_2 \) such that

\[
\|u\|_{H^{-1/2}(Q)}^2 \geq C \sum_{k=1}^{p_1} \sum_{l=1}^{p_2} u_{k,l}^2 \frac{1}{k^3 k^3}.
\]  

**Proof.** Due to \( \tilde{H}^{-1/2}(Q) = (H^{1/2}(Q))^t \) we have

\[
\|u\|_{\tilde{H}^{-1/2}(Q)} = \sup_{v \in H^{1/2}(Q)} \frac{(u, v)}{\|v\|_{H^{1/2}(Q)}}.
\]  

For \( v = \sum_{k=1}^{p_1} \sum_{l=1}^{p_2} v_{k,l} L_k L_l \) with \( v_{k,l} = u_{k,l}/(k^2 l^2) \) we therefore obtain

\[
\|u\|_{\tilde{H}^{-1/2}(Q)} \geq \frac{(u, v)}{\|v\|_{H^{1/2}(Q)}} = \sum_{k=1}^{p_1} \sum_{l=1}^{p_2} v_{k,l}^2 \frac{1}{k^2 l^2} \left( \frac{2k+1}{2l+1} \right) \frac{2}{k^3 l^3}.
\]  

Due to Lemma 6.10 we have

\[
\|v\|_{H^{1/2}(Q)}^2 \leq C \sum_{k=1}^{p_1} \sum_{l=1}^{p_2} v_{k,l}^2 \left( \frac{k}{l} + \frac{l}{k} \right) = C \sum_{k=1}^{p_1} \sum_{l=1}^{p_2} u_{k,l}^2 \frac{1}{k^3 l^3} \left( \frac{k}{l} + \frac{l}{k} \right) \leq C \sum_{k=1}^{p_1} \sum_{l=1}^{p_2} u_{k,l}^2 \frac{1}{k^3 l^3}.
\]

This completes the proof. \( \square \)

**Lemma 6.10** Let \( Q = (-1, 1)^2 \) and \( v = \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} v_{k,l} L_k L_l. \) Then there exists a constant \( C \) independent of \( v \) and \( p_1, p_2 \) such that

\[
\|v\|_{H^{1/2}(Q)}^2 \leq C \sum_{k=1}^{p_1} \sum_{l=1}^{p_2} v_{k,l}^2 \left( \frac{2k+1}{2l+1} + \frac{2l}{2k+1} \right).
\]  

**Proof.** From the definition of the interpolation norm \( \|v\|_{H^{1/2}(Q)} \) by the real K-method [9] we have

\[
\|v\|_{H^{1/2}(Q)}^2 = \|v\|_{L^2(Q), H^1(Q)}^2 = \int_0^\infty \left( t^{-1/2} \inf_{v = w + a} (\|w\|_{L^2(Q)} + t\|a\|_{H^1(Q)}) \right)^2 \frac{dt}{t}.
\]  

With \( w = \sum_{k=1}^{p_1} \sum_{l=1}^{p_2} u_{k,l} L_k L_l \) we have

\[
\|v\|_{H^{1/2}(Q)}^2 \leq \int_0^\infty \left( t^{-1/2} \inf_w (\|w\|_{L^2(Q)} + t\|w - v\|_{H^1(Q)}) \right)^2 \frac{dt}{t}
\]

\[
\leq 2 \int_0^\infty t^{-2} \inf_w (\|w\|_{L^2(Q)}^2 + t^2\|w - v\|_{H^1(Q)}^2) dt.
\]  

(6.46)
Due to Lemma 6.11 we have

\[
(L'_k, L'_m) = 2 \sum_{l=0}^{[m-1/2]} \sum_{n=0}^{[m-1/2]} (2(m - 2n) - 1) \delta_{k-1-2l,m-1-2n}.
\] (6.47)

Therefore we get

\[
\| \nabla (v' - w') \|_{L^2(Q)}^2 = \| \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} (v_{k,l} - w_{k,l}) L_k L_l \|_{L^2(Q)}^2 + \| \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} (v_{k,l} - w_{k,l}) L_k L'_l \|_{L^2(Q)}^2
\]

\[
= \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} \sum_{m=0}^{p_2} \sum_{n=0}^{p_2} (v_{k,l} - w_{k,l}) (v_{m,n} - w_{m,n}) ((L'_k, L'_m) (L_l, L_n) + (L_k, L_m) (L'_l, L'_n))
\]

\[
\leq 2 \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} (v_{k,l} - w_{k,l})^2 \left( \frac{2k^3}{2l+1} + \frac{2l^3}{2k+1} \right)
\] (6.48)

and we obtain

\[
\| v' \|_{H^{1/2}(Q)}^2 \leq 2 \int_0^t \inf_{w_{k,l}} \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} \left( \frac{2k^3}{2l+1} + \frac{2l^3}{2k+1} \right) dt
\]

\[
+ t^2 \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} (v_{k,l} - w_{k,l})^2 \left( \frac{2k^3}{2l+1} + \frac{2l^3}{2k+1} \right) dt
\]

\[
\leq 2 \int_0^t \inf_{w_{k,l}} \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} \left( \frac{2k^3}{2l+1} + \frac{2l^3}{2k+1} \right) dt
\]

\[
+ t^2 \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} (v_{k,l} - w_{k,l})^2 3 \left( \frac{2k^3}{2l+1} + \frac{2l^3}{2k+1} \right) dt
\]

\[
= 2 \int_0^t t^2 \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} \left( \frac{2k^3}{2l+1} + \frac{2l^3}{2k+1} \right) dt
\]

\[
= 2 \int_0^t \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} \frac{2k^3}{2l+1} + \frac{2l^3}{2k+1} dt
\]

\[
= \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} \frac{2k^3}{(2k+1)} + \frac{2l^3}{(2l+1)}
\]

\[
\leq \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} \frac{2k^2}{(2l+1)^2} + \frac{2l^2}{(2k+1)^2}
\]

\[
\leq \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} \frac{2k}{2l+1} + \frac{2l}{2k+1}
\] (6.49)
Lemma 6.11 Let \( k, m \geq 1 \). Then we have
\[
(L'_k, L'_m) = 2 \sum_{l=0}^{[(k-1)/2]} \sum_{n=0}^{[(m-1)/2]} (2(m - 2n) - 1) \delta_{k-1-2l,m-1-2n}.
\] (6.50)

Proof. From the recurrence formula of the Legendre polynomials we have
\[
L'_k - L'_{k-2} = (2k - 1) L_{k-1}.
\]
From this we obtain
\[
L'_k = \sum_{l=0}^{[(k-1)/2]} (2k - 2l - 1) L_{k-1-2l} \quad (6.51)
\]
and
\[
(L'_k, L'_m) = \sum_{l=0}^{[(k-1)/2]} (2k - 2l - 1) \sum_{n=0}^{[(m-1)/2]} (2(m - 2n) - 1)(L_{k-1-2l}, L_{m-1-2n})
\]
\[
= \sum_{l=0}^{[(k-1)/2]} (2k - 2l - 1) \sum_{n=0}^{[(m-1)/2]} (2(m - 2n) - 1) \frac{2 \delta_{k-1-2l,m-1-2n}}{2(k - 1 - 2l) + 1}
\]
\[
= 2 \sum_{l=0}^{[(k-1)/2]} \sum_{n=0}^{[(m-1)/2]} (2(m - 2n) - 1) \delta_{k-1-2l,m-1-2n}.
\]
\[\square\]

Lemma 6.12 Let \( Q = (-1, 1)^2 \) and \( u = \sum_{k=2}^{p_1} \sum_{l=2}^{p_2} u_{k,l} L_k L_l, 2 \leq p_1, 2 \leq p_2 \). Then there exists a constant \( C \) independent of \( u, p_1, p_2 \) such that
\[
\|u\|^2_{H^{1/2}(Q)} \geq C \sum_{k=2}^{p_1} \sum_{l=2}^{p_2} u_{k,l}^2 \left( \frac{1}{k^4} + \frac{1}{l^4} \right).
\] (6.52)

Proof. From the definition of the interpolation norm \( \| \cdot \|_{H^{1/2}(Q)} \) by the real K-method [9] we have
\[
\|u\|^2_{H^{1/2}(Q)} = \|u\|^2_{[L^2(Q), H^1_0(Q)]_{1/2}} = \int_0^\infty \left( t^{-1/2} \inf_{u=v+w \text{ with } w \in H^1_0(Q)} (\|v\|_{L^2(Q)} + t\|w\|_{H^1_0(Q)}) \right)^2 \frac{dt}{t}.
\] (6.53)
We have \( u = \sum_{k=2}^{p_1} \sum_{l=2}^{p_2} u_{k,l} L_k L_l \in H^1_0(Q) \). Therefore \( u = v + w \) with \( w \in H^1_0(Q) \) implies also \( v \in H^1_0(Q) \). Therefore it is sufficient to take the infimum in \( H^1_0(Q) \).
\[
\|u\|^2_{H^{1/2}(Q)} = \int_0^\infty t^{-2} \left( \inf_{v \in H^1_0(Q)} (\|v\|_{L^2(Q)} + t\|u - v\|_{H^1_0(Q)}) \right)^2 dt.
\] (6.54)
Due to \( v \in H^1_0(Q) \) we can expand \( v \) also in antiderivatives of Legendre polynomials
\[
v = \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} v_{k,l} L_k L_l.
\]
Using the orthogonality of the Legendre polynomials and Lemma 6.13 we get for the norms in (6.54)

\[ \| u - v \|_{H^1_0(Q)}^2 = \| \partial_x (u - v) \|_{L^2(Q)}^2 + \| \partial_y (u - v) \|_{L^2(Q)}^2 \]
\[ = \left( \sum_{k=2}^{\infty} \sum_{l=2}^{2k-1} (u_{k,l} - v_{k,l}) L_{k-1} L_l \right)^2_{L^2(Q)} + \left( \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} (u_{k,l} - v_{k,l}) L_k L_{l-1} \right)^2_{L^2(Q)} \]
\[ = \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{2}{2k-1} (u_{k,l} - v_{k,l})^2 \frac{1}{l^5} + \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{2}{2l-1} (u_{k,l} - v_{k,l})^2 \frac{1}{k^5} \]
\[ = C \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} (u_{k,l} - v_{k,l})^2 \frac{2}{2k-1} \frac{1}{l^5} + C \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} (u_{k,l} - v_{k,l})^2 \frac{2}{2l-1} \frac{1}{k^5}. \quad (6.55) \]

Applying Lemma 6.13 twice we obtain

\[ \| v \|_{L^2(Q)}^2 \geq C \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} v_{k,l}^2 k^{-5} l^{-5}. \quad (6.56) \]

Using the monotonicity of the square function and the inequality \((a+b)^2 \leq 2(a^2 + b^2)\) we get from (6.54)

\[ \| u \|_{H^{1/2}_0(Q)}^2 = \int_0^{\infty} t^{-2} \left( \inf_{v \in H^1_0(Q)} \left( \| v \|_{L^2(Q)} + t \| u - v \|_{H^1_0(Q)} \right) \right)^2 dt \]
\[ = \int_0^{\infty} t^{-2} \inf_{v \in H^1_0(Q)} \left( \| v \|_{L^2(Q)} + t \| u - v \|_{H^1_0(Q)} \right)^2 dt \]
\[ \geq \int_0^{\infty} t^{-2} \inf_{v \in H^1_0(Q)} \left( \| v \|_{L^2(Q)}^2 + t^2 \| u - v \|_{H^1_0(Q)}^2 \right) dt \]
\[ \geq \int_0^{\infty} t^{-2} \inf_{v \in H^1_0(Q)} \left( \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \left( v_{k,l}(kl)^{-5} + t^2 (u_{k,l} - v_{k,l})^2 \left( \frac{2}{2k-1} \frac{1}{l^5} + \frac{2}{2l-1} \frac{1}{k^5} \right) \right) \right) dt \]
\[ = C \sum_{k=2}^{p_1} \sum_{l=2}^{p_2} \int_0^{\infty} t^{-2} \inf_{v \in H^1_0(Q)} \left( v_{k,l}(kl)^{-5} + t^2 (u_{k,l} - v_{k,l})^2 \left( \frac{2}{2k-1} \frac{1}{l^5} + \frac{2}{2l-1} \frac{1}{k^5} \right) \right) dt \quad (6.57) \]

If we investigate the minimum of the polynomial \(f(x) = ax^2 + b(x - u)^2\) we get from \(f'(x_0) = 2ax_0 + 2b(x_0 - u) = 0\) that we have \(x_0 = \frac{2b}{a+b}\) and \(f(x_0) = u^2 \frac{ab}{a+b}\). Therefore
we get from equation (6.57)

\[ \| u \|_{H^{1/2}(Q)}^2 \geq C \sum_{k=2}^{P_1} \sum_{l=2}^{P_2} \int_0^\infty \! t^{-2} u_{k,l}^2 \frac{t^2(kl)^{-5} \left( \frac{2}{2k-1} + \frac{2}{2l-1} \frac{1}{k^5} \right)}{(kl)^{-5} + t^2 \left( \frac{2}{2k-1} + \frac{2}{2l-1} \frac{1}{k^5} \right)} \]

\[ = C \sum_{k=2}^{P_1} \sum_{l=2}^{P_2} \int_0^\infty \! u_{k,l}^2 \frac{\pi \sqrt{(kl)^{-5} \left( \frac{2}{2k-1} + \frac{2}{2l-1} \frac{1}{k^5} \right)} \left( \frac{2}{2k-1} + \frac{2}{2l-1} \frac{1}{k^5} \right) \sqrt{\left( (kl)^{-5} \left( \frac{2}{2k-1} + \frac{2}{2l-1} \frac{1}{k^5} \right) \right)}}{2} \]

\[ \geq C \sum_{k=2}^{P_1} \sum_{l=2}^{P_2} \frac{u_{k,l}^2 \pi}{2} \left( \frac{1}{k^5} + \frac{1}{l^5} \right). \quad (6.58) \]

This completes the proof. \( \square \)

**Lemma 6.13** Let \( I = [-1, 1] \) and \( v = \sum_{k=2}^\infty v_k \mathcal{L}_k \). Then, there holds

\[ \| v \|_{L^2(I)}^2 \geq C \sum_{k=2}^\infty \frac{v_k^2}{k^5}. \quad (6.59) \]

**Proof.** With the normalized antiderivatives of Legendre polynomials defined by \( \mathcal{L}_k^* = \mathcal{L}_k/\| \mathcal{L}_k \|_{L^2(I)} \), there holds due to the proof of [55, Theorem 1]

\[ \| v \|_{L^2(I)}^2 = \left\| \sum_{k=2}^\infty v_k \mathcal{L}_k \|_{L^2(I)} \right\|_{L^2(I)}^2 \geq C \sum_{k=2}^\infty \frac{v_k^2 k^{-2} \| \mathcal{L}_k \|_{L^2(I)}^2}{k^5}. \quad (6.60) \]

Due to

\[ \| \mathcal{L}_k \|_{L^2(I)}^2 = \left\| \frac{1}{2k-1} (L_k - L_{k-2}) \right\|_{L^2(I)}^2 \]

\[ = \left( \frac{1}{2k-1} \right)^2 \left( \frac{2}{2k+1} + \frac{2}{2k-3} \right) \]

\[ = \frac{4}{(2k+1)(2k-1)(2k-3)} \geq C \frac{1}{k^3} \quad (k \geq 2) \quad (6.61) \]

we get

\[ \| v \|_{L^2(I)}^2 \geq C \sum_{k=2}^\infty \frac{v_k^2}{k^5}. \quad (6.62) \]

This completes the proof. \( \square \)
Chapter 7

Approximation in countably normed spaces

In this chapter we analyze the approximation of functions of countably normed spaces in the $H^{1/2}$ norm. This is a generalization of the techniques introduced in [46] to edge-oriented weight-functions, anisotropic meshes and non-integer Sobolev-spaces. The countably normed space $B^1_{\bar{\beta}}$ is appropriate to describe the solutions of the hypersingular integral equation in three dimensions, see [57].

7.1 Introduction

This chapter extends the series of papers [89, 70, 58, 53, 57, 71] on the $hp$ version of the boundary element method for three dimensional problems. Approximation results for the $hp$ version of the finite element method for elliptic boundary value problems were obtained by Babuška and Guo (see [46] for 2D and [6, 5, 45] for 3D problems). In [70] we prove that a function $u \in B^1_{\bar{\beta}}(Q)$, $0 \leq \beta < 1$, can be approximated with an exponentially fast convergence rate in the $L^2$ norm by appropriately chosen piecewise polynomials on a geometric mesh on $Q = [0,1] \times [0,1]$, where $B^l_{\bar{\beta}}(Q)$, $l$ integer, is a countably normed space. In [71] we prove that the exponential convergence also holds in the $H^1(Q)$ norm if $u \in B^1_{\bar{\beta}}(Q)$ with $0 \leq \beta < 1$. Here we show that the exponential convergence also holds in the $H^{1/2}(Q)$ norm if $u \in B^1_{\bar{\beta}}(Q)$ with $0 \leq \beta < 1/2$. This convergence result in the $H^{1/2}$ norm is not a straightforward generalization of the $L^2$ error estimate since the approximation in the $H^{1/2}$ norm for $B^1_{\bar{\beta}}$ functions needs a much more refined analysis due to the continuity requirement for the approximation on the element boundaries. If $u$ vanishes on the boundaries we show also exponentially fast convergence in the $\tilde{H}^{1/2}(Q)$ norm.

As a model problem we consider a hypersingular integral equation which results from the Neumann boundary value problem for the Laplacian in a polyhedral domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma$. The approximation results developed below can be applied to the Lamé and Helmholtz equations as well. The boundary integral equation under consideration here is

\[ Wv = (I - K')g \text{ on } \Gamma. \tag{7.1} \]

The hypersingular integral operator and the adjoint double layer potential are given by

\[ Ww(x) := -2 \partial_{n_x} \int_{\Gamma} \frac{\partial}{\partial n_x} G(x, y) w(y) \, ds_y, \quad K'w(x) := 2 \int_{\Gamma} \frac{\partial}{\partial n_x} G(x, y) w(y) \, ds_y \]
with \( G(x, y) = \frac{1}{4\pi} |x - y|^{-1} \). Here \( v = u|_{\Gamma} \) and \( u \) solves the boundary value problem:

For given \( g \in H^{-1/2}(\Gamma) \) with \( \int_{\Gamma} g \, ds = 0 \) find \( u \in H^1(\Omega)/\mathbb{R} \) such that

\[
\triangle u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma. \tag{7.2}
\]

with the normal vector \( n \) always pointing away from \( \Omega \). For the definitions of the Sobolev spaces \( H^k(\Omega) \), \( k \) integer, and \( H^s(\Gamma) \), \( s \in \mathbb{R} \), see [66]. It is known from [98] that for \( g \in H^{-1/2}(\Gamma) \) with \( \int_{\Gamma} g \, ds = 0 \) there exists exactly one solution \( v \in H^{1/2}(\Gamma)/\mathbb{R} \) of the hypersingular integral equation (7.1). Due to [98] the Neumann problem (7.2) and the integral equation (7.1) are equivalent; i.e., let \( u \in H^1(\Omega) \) solve (7.2) then \( v = u|_{\Gamma} \) solves (7.1), and conversely, let \( v \in H^{1/2}(\Gamma) \) solve (7.1) then \( u \) defined by the representation formula with \( u|_{\Gamma} := v \) and \( \frac{\partial u}{\partial n} := g \) solves (7.2).

The integral equation (7.1) can be solved approximately by Galerkin’s method using subspaces \( \{X_N\} \) of \( H^{1/2}(\Gamma)/\mathbb{R} \), i.e.: for given \( g \in H^{-1/2}(\Gamma) \) with \( \int_{\Gamma} g \, ds = 0 \) find \( v_N \in X_N \) such that

\[
\langle Wv_N, w \rangle = \langle (I - K^\ast)g, w \rangle \quad \forall w \in X_N. \tag{7.3}
\]

Here the brackets \( \langle \cdot, \cdot \rangle \) denote the duality between the Sobolev spaces \( H^{1/2}(\Gamma) \) and \( H^{-1/2}(\Gamma) \) which can be identified with the \( L^2 \) inner product for smooth functions.

Due to [80] the hypersingular integral operator \( W \) in \( \langle Wv_N, w \rangle \) can be reduced by partial integration to the weakly singular integral operator \( V \) applied to surface gradients of test and trial functions, i.e.,

\[
\langle Wv_N, w \rangle = \langle V \text{curl}_\Gamma v_N, \text{curl}_\Gamma w \rangle \quad \forall v_N, w \in X_N,
\]

using \( \text{curl}_\Gamma = \mathbf{n} \times \nabla_{\Gamma} \).

Since the bilinear form \( \langle \cdot, \cdot \rangle \) is strongly elliptic in \( H^{1/2}(\Gamma)/\mathbb{R} \) (cf. Theorem 6 in Chapter XI of [31]), the unique boundary element Galerkin solution \( v_N \in X_N \) converges quasi-optimally in \( H^{1/2}(\Gamma) \); i.e., for all \( N \)

\[
\|v - v_N\|_{H^{1/2}(\Gamma)} \leq C \inf_{w \in X_N} \|v - w\|_{H^{1/2}(\Gamma)} \tag{7.4}
\]

with a constant \( C \) which is independent of \( N \), the Galerkin solution \( v_N \) and the exact solution \( v \).

If \( \Gamma \) is an open surface, then the boundary integral equation reads

\[
Wv = g \text{ on } \Gamma. \tag{7.5}
\]

The integral equation (7.5) can be solved approximately by Galerkin’s method using subspaces \( \{X_N\} \) of \( \tilde{H}^{1/2}(\Gamma) \), i.e.: for given \( g \in H^{-1/2}(\Gamma) \) find \( v_N \in X_N \) such that

\[
\langle Wv_N, w \rangle = \langle g, w \rangle \quad \forall w \in X_N. \tag{7.6}
\]

The unique boundary element Galerkin solution converges optimally in \( \tilde{H}^{1/2}(\Gamma) \); i.e., for all \( N \)

\[
\|v - v_N\|_{\tilde{H}^{1/2}(\Gamma)} \leq C \inf_{w \in X_N} \|v - w\|_{\tilde{H}^{1/2}(\Gamma)} \tag{7.7}
\]

with a constant \( C \) which is independent of \( N \), the Galerkin solution \( v_N \) and the exact solution \( v \) of (7.5).

The quasi-optimality estimates (7.4) and (7.7) show that the rates of convergence of the Galerkin solutions are determined by the choice of the approximating subspace \( X_N \).
and the regularity of the exact solution $v$ of (7.1) or (7.5), respectively. There are three versions of the boundary element method. The classical $h$ version, which achieves the accuracy by refining the mesh while using low-degrees $p$ of elements, usually $p = 1$ or 2. The $p$ version keeps the mesh fixed and the accuracy is achieved by increasing $p$. The $hp$ version, which is considered in this chapter, contains both approaches. The $h$ version of the Galerkin schemes (7.3) and (7.6) with mesh grading towards the edges of the polyhedron $\Gamma$ are considered in [98], whereas for the $p$ version, see [86].

Here we show the exponentially fast convergence of the boundary element Galerkin solution for the hypersingular integral equation (7.1) on a polyhedral surface $\Gamma$ and for the hypersingular integral equation (7.5) on an open surface $\Gamma$ when the $hp$ version on a geometric mesh refinement $\Gamma^\delta$ is performed. We call our mesh a tensor product mesh since it is locally the affine image of a geometric mesh on the reference element obtained by refinement towards the coordinate axes. This is easily implemented and differs from the geometric mesh introduced by Guo and Babuška [5].

**Theorem 7.1** [57, Theorem 2 and Remark 2] Let $v_1$ be the solution of (7.1). Then there holds

$$v_1 \in \mathcal{B}_\beta^{1}(\Gamma) \cap C^0(\Gamma) \text{ with } 0 \leq \beta < 1/2.$$  

Let $v_2$ be the solution of (7.5). Then there holds

$$v_2 \in \mathcal{B}_\beta^{1}(\Gamma) \cap C^0(\Gamma) \text{ with } 0 \leq \beta < 1/2 \text{ and } v_2 \big|_{\partial \Gamma} = 0.$$  

**Corollary 7.1** Due to Lemma 7.13 we have $v_1 \in H^{1/2}(\Gamma)$. Due to Lemma 7.13 and Lemma 7.15 we have $v_2 \in \tilde{H}^{1/2}(\Gamma)$.

For results on the regularity of $v$ under conditions on the data, see [57, 53], where we show that the boundary terms of the solutions of Dirichlet and Neumann problems for the Laplacian can be separated into parts in countably normed spaces plus remainder terms of arbitrarily high regularity. As shown in [53] such remainder terms can be approximated by the $hp$ version of arbitrary high order. Therefore, this work is concentrated on the approximation of functions from countably normed spaces in the $H^{1/2}(\Gamma)$ norm, the energy norm of the hypersingular integral operator. For further regularity results see also [43], [44].

Now, we define the geometric mesh $\Gamma^\delta$ on the faces of the polyhedron. We assume that the face $F$ is a triangle. This is no loss of generality because every polygonal domain can be decomposed into triangles. We divide this triangle into three parallelograms and three triangles where each parallelogram lies in a corner of the face $F$ and each triangle lies at an edge of $F$ apart of the corners. By linear transformations $\varphi_i$ we can transform the parallelograms onto the reference square $Q = [0,1]^2$ such that the vertices of the face $F$ are transformed to $(0,0)$. The triangles can be transformed by a linear transformation $\tilde{\varphi}_i$ on to the reference triangle $\tilde{Q} = \{ (x,y) \in Q \mid y \leq x \}$ such that the corner point of the triangle in the interior of the face $F$ is transformed to $(1,1)$ of the reference triangle. By Definition 7.1, the geometric mesh and appropriate spline spaces are defined on the reference element $Q$. Analogously the geometric mesh can be defined on the reference triangle $\tilde{Q}$ (see Figure 7.1). Via the transformations $\varphi_i^{-1}, \tilde{\varphi}_i^{-1}$ the geometric mesh is also defined on the faces of the polyhedron. The approximation on the reference square is the more interesting case because it handles the corner-edge singularities. Therefore, we deal in this paper only with the approximation on the reference square.

Let $\Gamma$ be the boundary of a simply connected bounded polyhedron $\Omega$ in $\mathbb{R}^3$. We introduce appropriate countably normed spaces $\mathcal{B}_\beta^{1}(\Gamma), (0 < \beta < 1)$ (see also [35]). Let $F$ be
a face of $\Omega$ having the corner points $e_1, \ldots, e_m$. For each $i$, $f_i$ denotes the edge of $F$ connecting $e_i$ with $e_{i+1}$, where the periodicity convention $e_{m+1} = e_1, f_{m+1} = f_1$ is adopted. Consider a covering of $F$ by neighborhoods $U_i$ ($i = 1, \ldots, m$) of $e_i$ not containing $e_j, j \neq i$, and introduce polar coordinates $(r_i, \Theta_i)$ of origin $e_i$ in $U_i$ such that $r_i = \text{dist}(x, e_i)$ and the edges $f_{i-1}$ and $f_i$ are given by $\Theta_i = \omega_i$ and $\Theta_i = 0$ respectively. For $0 < \beta < 1$, let

$$B_\beta^I(F) = \{u | u \in H^{-1}(F) : \exists C, d > 0, \text{ independent of } k \text{ such that}$$

$$\|v_\beta^{\beta+\alpha_\gamma-1}((\Theta_i - \Theta_j))((\beta+\alpha_\gamma-\gamma) + (\partial/\partial r_i)\alpha_r (\partial/\partial \Theta_i)\alpha_\gamma u\|_{L^2(U_i)} \leq$$

$$\leq C^d \alpha_r + \alpha_\gamma = k = l, l + 1, \ldots, i = 1, \ldots, m\},$$

$$B_\beta^I(\Gamma) = \{u | u \in H^{-1}(\Gamma), u \in B_\beta^I(F) \text{ for all faces } F \text{ of } \Gamma\}.$$ 

with $(a)_+ := \max(a, 0)$.

**Theorem 7.2** Let $v$ be the solution of (7.5) and $v_N \in X_N = S^{p-1}(\Gamma_n)$ its Galerkin approximation from (7.6) where $S^{p-1}(\Gamma_n)$ is the space of continuous, piecewise polynomials on $\Gamma_n$. Then there holds

$$\|v - v_N\|_{H^{1/2}(\Gamma)} \leq C e^{-b\sqrt{N}}$$

for constants $C, b > 0$ depending on the mesh parameters $\sigma, \mu, \beta$ but not on $N := \dim X_N$.

**Proof.** Due to Theorem 7.1 we have that $v \in B_\beta^I(\Gamma)$ with $v|_{\partial \Gamma} = 0$ and $0 \leq \beta < 1/2$. Using Theorem 7.3 we can show by an affine mapping to the reference element, that we have locally, i.e., near to the corners, $v \in B_\beta^I(Q)$ with $v|_{[0,1] \times \{0\}} = v|_{\{0\} \times [0,1]}$ and $0 \leq \beta < 1/2$. Due to Theorem 7.5 we have exponentially fast convergence on the reference element. Patches near edges, with a finite distance to the corners (cf. Figure 7.1), have to take into account only edge singularities. The approximation can be done by using one-dimensional results.

The exponentially fast approximation on the triangular elements in Figure 7.1 can be proven similar to the rectangular elements. The mesh on the reference triangle consists mainly of anisotropic rectangles. The triangles involved are always shape-regular, therefore, standard approximation theorems can be applied. 

**Definition 7.1 (geometric mesh)** Let $Q = [0,1] \times [0,1]$. For $0 < \sigma < 1$ we use the partition $Q^n_\sigma$ of $Q$ into $n^2$ sub-squares $R_{kl}$

$$R_{kl} = [x_{k-1}, x_k] \times [x_{l-1}, x_l], \quad (k, l = 1, \ldots, n), \quad Q = \bigcup_{k,l=1}^n R_{kl}$$

where

$$x_0 = 0, \quad x_k = \sigma^{n-k}, \quad k = 1, \ldots, n. \quad (7.10)$$

With $Q^n_\sigma$ we associate a degree vector $p = (p_1, \ldots, p_n)$ and define $S^{p-1}(Q^n_\sigma) \subset C^0(Q)$ as the vector space of all piecewise polynomials $v(x, y)$ on $Q$ having degree $p_k$ in $x$ and $p_l$ in $y$ on $[x_{k-1}, x_k] \times [x_{l-1}, x_l], k,l = 1, \ldots, n$, i.e., $v|_{[x_{k-1}, x_k] \times [x_{l-1}, x_l]} \in P_{p_k, p_l}(R_{kl})$. For the differences $h_k = x_k - x_{k-1}$ we have with $\lambda = (1 - \sigma)/\sigma$

$$h_k = x_k - x_{k-1} = x_{k-1}(1 - \sigma) \leq x(1 - \sigma) = x\lambda, \quad \forall x \in [x_{k-1}, x_k], \quad 2 \leq k \leq n. \quad (7.12)$$
Let an integer $\bar{\nu} < 0$

Now we define the countably normed spaces on the reference element

Definition 7.3 (weighted Sobolev spaces)

The weighted Sobolev spaces are defined for integers $\nu$ and $\nu'$

Due to [66, Theorem 11.7] the space $\tilde{H}^{1/2}(\Gamma)$, where $\Gamma$ is an open surface, can be characterized by

$$\tilde{H}^{1/2}(\Gamma) = \{ v \in H^{1/2}(\Gamma) : \varrho^{-1/2} v \in L^2(\Gamma) \}, \quad \varrho(x) := \text{dist}(x, \partial \Gamma)$$

and

$$\|v\|_{\tilde{H}^{1/2}(\Gamma)}^2 = \|v\|_{H^{1/2}(\Gamma)}^2 + \int_{\Gamma} \frac{|v|^2}{\varrho}$$

defines a norm in $\tilde{H}^{1/2}(\Gamma)$.

This leads to the following definition on the reference element

Definition 7.2 Let $\varrho(x,y) := \min\{x,y\}$. Then we write

$$\tilde{H}^{1/2}(Q) := \{ v \in H^{1/2}(Q) : \varrho^{-1/2} v \in L^2(Q) \}$$

and

$$\|v\|_{\tilde{H}^{1/2}(Q)}^2 = \|v\|_{H^{1/2}(Q)}^2 + \int_0^1 \int_0^1 \frac{|v(x,y)|^2}{\varrho(x,y)} dy dx.$$  \(7.14\)

Now we define the countably normed spaces on the reference element $Q$ using Cartesian coordinates which we need to prove the approximation results.

Definition 7.3 (weighted Sobolev spaces $H^{m,l}_{\beta}(Q)$)

Let $\beta$ be a real number with $0 < \beta < 1$. The weight function $\Phi_{\beta,\alpha,l}(x,y)$ is for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ and an integer $l \geq 1$ defined by

$$\Phi_{\beta,\alpha,l} = \sum_{\gamma_1 = \max(\alpha_1 - l, 0)}^{\min(\alpha_1, \alpha_2 - l)} x^{\gamma_1} y^{\alpha_1 + \alpha_2 - l - \gamma_1} + \sum_{\gamma_2 = \max(\alpha_2 - l, 0)}^{\min(\alpha_2, \alpha_1 - l)} x^{\alpha_1 + \alpha_2 - l - \gamma_2} y^{\gamma_2}.$$  \(7.15\)

Let

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} = \partial_x^{\alpha_1} \partial_y^{\alpha_2}.$$

The weighted Sobolev spaces are defined for integers $m$ and $l$, $m \geq l \geq 1$ by

$$H^{m,l}_{\beta}(Q) = \left\{ u : u \in H^{l-1}(Q) \text{ for } l > 0, \|\Phi_{\beta,\alpha,l} D^\alpha u\|_{L^2(Q)} < \infty \text{ for } l \leq |\alpha| \leq m \right\},$$  \(7.16\)
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with the norm

\[ \|u\|_{H^{m,l}_\beta(Q)}^2 = \|u\|_{H^{2-1}(Q)}^2 + \sum_{k=|\alpha|=l}^m \int_Q |D^\alpha u(x)|^2 \Phi_{\beta,\alpha,l}(x) \, dx \]  

(7.17)

and the semi-norm

\[ |u|_{H^{m,l}_\beta(Q)}^2 = \sum_{k=|\alpha|=l}^m \int_Q |D^\alpha u(x)|^2 \Phi_{\beta,\alpha,l}(x) \, dx. \]  

(7.18)

Definition 7.4 (countably normed spaces \(B^l_{\beta}(Q)\))

The countably normed spaces for \(l \geq 1\) are defined by

\[ B^l_{\beta}(Q) = \left\{ u : u \in H^{k,l}_\beta(Q) \forall k \geq l, \|\Phi_{\beta,\alpha,l}D^\alpha u\|_{L^2(Q)} \leq C d^{k-l}(k - l)! \right\} \]  

for \(|\alpha| = k = l, l + 1, \ldots, \) with \(C \geq 1, d \geq 1\) independent of \(k\) \(\) \(\)  

(7.19)

If we want to emphasize the dependence on the constants \(C, d\) we will write \(B^l_{\beta,C,d}(Q)\), etc.

Theorem 7.3 [67] Let \(Q\) be the reference element and let \(\varphi\) be the linear transformation from a parallelogram, which lies in a corner of the face \(F\), to the reference element \(Q\). Then

\[ u \circ \varphi^{-1} \in B^1_{\beta,C,d}(Q) \]  

(7.20)

if and only if

\[ u \in B^1_{\beta,C,d}(\varphi(Q)) \]  

(7.21)

where \(C, d\) (resp. \(\tilde{C}, \tilde{d}\)) are the constants in the definition of \(B^1_{\beta}(Q)\) (resp. \(B^1_{\beta}(\varphi(Q))\)).

Corollary 7.2 For \(l = 1\) we have the following weight function from (7.15)

\[ \Phi_{\beta,(\alpha_1,\alpha_2),l} = \begin{cases} x^{\beta+\alpha_1-1}, & \alpha_1 \geq 1, \alpha_2 = 0 \\ x^{\beta+\alpha_1-1}y^{\alpha_2} + x^{\alpha_1}y^{\beta+\alpha_2-1}, & \alpha_1 \geq 1, \alpha_2 \geq 1 \\ y^{\beta+\alpha_2-1}, & \alpha_1 = 0, \alpha_2 \geq 1 \end{cases} \]  

(7.22)

The weighted Sobolev spaces \(H^{k,l}_\beta(Q)\) are defined only for positive integers \(k \geq l > 0\).

For the proof of the approximation theorems we need also the weighted Sobolev spaces \(H^{s,l}_\beta(Q)\) of a non-integral \(s\) which are defined as interpolation spaces by the \(K\)-method (see [9]):

\[ (H^{k,l}_\beta(Q), H^{k+1,l}_\beta(Q))_{\theta,\infty} = H^{k+\theta,l}_\beta(Q), \quad 0 < \theta < 1. \]  

It can be shown by interpolation [51] that if \(u \in B^l_{\beta,C,d}(Q)\), then for any \(k \geq l\)

\[ |u|_{H^{k+\theta,l}_\beta(Q)} \leq Cd^{k+\theta-l}(k + \theta - l + 1), \quad 0 < \theta < 1. \]  

(7.23)
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7.2 Exponential convergence

**Theorem 7.4** Let $u(x, y) \in B^1_\beta(Q)$ with $0 \leq \beta < 1/2$. Then there is a spline function $u_N \in S^{p, 1}(Q^n)$ and constants $C, b > 0$ independent of $N$, but dependent on $\sigma, \mu, \beta$ such that

$$\|u(x, y) - u_N(x, y)\|_{H^{1/2}(Q)} \leq C e^{-b \sqrt[N]}$$

(7.24)

with $p = (p_1, \ldots, p_n)$, $p_1 = 1$, $p_k = \max(2, \lceil \mu(k - 1) \rceil + 1)$ ($k > 1$) for $\mu > 0$ and $N = \dim S^{p, 1}(Q^n)$.

**Proof.** See section 7.4.

**Theorem 7.5** Let $u(x, y) \in B^1_\beta(Q)$ with $0 \leq \beta < 1/2$ and $u|_{[0,1] \times \{0\}} \equiv 0$, $u|_{\{0\} \times [0,1]} \equiv 0$. Then there is a spline function $u_N \in S^{p, 1}(Q^n)$ and constants $C, b > 0$ independent of $N$, but dependent on $\sigma, \mu, \beta$ such that

$$\|u(x, y) - u_N(x, y)\|_{H^{1/2}(Q)} \leq C e^{-b \sqrt[N]}$$

(7.25)

with $p = (p_1, \ldots, p_n)$, $p_1 = 1$, $p_k = \max(2, \lceil \mu(k - 1) \rceil + 1)$ ($k > 1$) for $\mu > 0$ and $N = \dim S^{p, 1}(Q^n)$.

**Proof.** See section 7.5.

7.3 Approximation results

In this section we collect results necessary for the proof of the main theorem 7.4.

**Lemma 7.1** Let $Q = [0,1]^2$, $0 \leq \beta < 1/2$ and $u \in H^{2, 1}_\beta(Q)$. Then there holds for all $(x_1, x_2), (y_1, y_2) \in Q$

$$|u(x_1, x_2) - u(y_1, y_2)| \leq \frac{C}{1 - 2\beta} \left| \frac{y_1^{1-2\beta} - x_1^{1-2\beta}}{\min(x_1, y_1)^{1-\beta}} + \frac{y_2^{1-2\beta} - x_2^{1-2\beta}}{\min(x_2, y_2)^{1-\beta}} \right| u\|_{H^{2, 1}_\beta(Q)}$$

(7.26)

which implies

$$H^{2, 1}_\beta(Q) \subset C^0(Q \setminus \{(0,0)\})$$

**Proof.** For $u \in C^\infty(Q)$ and $(x_1, x_2), (y_1, y_2) \in Q$ with $x_1 \leq y_1$, $x_2 \leq y_2$ we write

$$u(x_1, x_2) - u(y_1, y_2) = \int_{x_1}^{y_1} \int_{x_2}^{y_2} \partial_w \partial_z u(w, z) \, dz \, dw - \int_{x_1}^{y_1} \partial_w u(w, y_2) \, dw - \int_{x_2}^{y_2} \partial_z u(y_1, z) \, dz.$$

Firstly, we estimate

$$\left( \int_{x_1}^{y_1} \int_{x_2}^{y_2} \partial_w \partial_z u(w, z) \, dz \, dw \right)^2 \leq \int_{x_1}^{y_1} \int_{x_2}^{y_2} (w^{\beta} z + wz^{\beta})^{-2} \, dz \, dw \int_{x_1}^{y_1} \int_{x_2}^{y_2} \left( (w^{\beta} z + wz^{\beta}) \partial_w \partial_z u(w, z) \right)^2 \, dz \, dw$$

and

$$\int_{x_1}^{y_1} \int_{x_2}^{y_2} (w^{\beta} z + wz^{\beta})^{-2} \, dz \, dw = \int_{x_1}^{y_1} \int_{x_2}^{y_2} w^{-2\beta} z^{-2\beta} (z^{1-\beta} + w^{1-\beta})^{-2} \, dz \, dw$$

$$\leq \frac{y_1^{1-2\beta} - x_1^{1-2\beta}}{1 - 2\beta} \frac{y_2^{1-2\beta} - x_2^{1-2\beta}}{1 - 2\beta} \frac{1}{(x_1^{1-\beta} + x_2^{1-\beta})^2}$$

$$< \infty \text{ for } (x_1, x_2) \neq (0,0).$$
Next, we write
\[
\int_{x_2}^{y_2} \partial_z u(y_1, z) \, dz = 2 \int_{1/2}^{y_1} \int_{x_2}^{y_2} \left( \partial_z u(y_1, z) - \partial_z u(w, z) + \partial_z u(w, z) \right) \, dz \, dw
\]
and further estimate
\[
\left( \int_{1/2}^{y_1} \int_{x_2}^{y_2} \partial_z \partial_z u(v, z) \, dz \, dv \right)^2 \leq \left( \int_{1/2}^{y_1} \int_{x_2}^{y_2} |\partial_z \partial_z u(v, z)| \, dz \, dv \right)^2
\]
Furthermore
\[
\left( \int_{1/2}^{y_1} \int_{x_2}^{y_2} \partial_z u(w, z) \, dz \, dw \right)^2 \leq \int_{1/2}^{y_1} \int_{x_2}^{y_2} z^{-2\beta} \, dz \, dw \int_{1/2}^{y_2} \int_{x_2}^{y_2} (z^\beta \partial_z u(w, z))^2 \, dz \, dw
\]
By a standard density argument inequality (7.26) holds for \( u \in H_\beta^{2,1}(Q) \). \( \square \)

Let
\[
\tilde{u}_{h_2}(x, y) := u(x, b_2) - \left(1 - \frac{y - a_2}{h_2}\right) \int_{a_2}^{b_2} \partial_z u(x, z) \, dz \tag{7.27}
\]
and
\[
G(\alpha_1) := \left( \frac{h_1}{2} \right)^{2(k_1 + 1 - \alpha_1)} \frac{(k_1 - \tilde{k}_1)!}{(k_1 + \tilde{k}_1 + 2 - 2\alpha_1)!}. \tag{7.28}
\]

**Lemma 7.2** (c.f. [67, Lemma 3.6]) Let \( Q_1 = (a_1, b_1) \times (a_2, b_2) \subseteq Q = [0, 1]^2 \) with \( h_1 = b_1 - a_1, h_2 = b_2 - a_2 > 0 \). For \( u \) sufficiently differentiable, there exist a polynomial \( \phi(x, y) = \sum_{0 \leq i \leq k_1, 0 \leq j \leq 1} d_{ij} x^i y^j \) on \( Q_1 \) with \( 1 \leq k_1 \leq k \) and a function \( \tilde{u}_{h_2}(x, y) \) such that for
\[
0 \leq |\alpha| = \alpha_1 + \alpha_2 \leq 1 \text{ and } 0 \leq \tilde{k}_1 \leq k_1 \text{ there holds with } D^\alpha = \partial_x^{\alpha_1} \partial_y^{\alpha_2}
\]

\[
\|D^\alpha(\tilde{u}_{h_2} - \phi)\|_{L^2(Q_1)}^2 \leq C G(\alpha_1) \left\{ \left\| \partial_x^{k_1+1} u(x, y) \right\|_{L^2(Q_1)}^2 + h_2^{-2\alpha_2} \left\| \partial_x^{k_1+1} \int_{a_2}^{b_2} \partial_z u(x, z) \, dz \right\|_{L^2(Q_1)}^2 \right\} \tag{7.29}
\]
and for \( z = a_2 \) or \( z = b_2 \)
\[
\| \partial_x^{\alpha_1} (u(x, z) - \phi(x, z)) \|_{L^2(a_1, b_1)}^2 \leq C G(\alpha_1) \left\{ \frac{1}{h_2} \left( \| \partial_x^{k_1 + 1} u(x, y) \|_{L^2(Q_1)}^2 \right) + \left\| \int_{a_2}^{b_2} \partial_z u(x, z) \, dz \right\|_{L^2(Q_1)}^2 \right\} \cdot (7.30)
\]

The constant \( C > 0 \) is independent of \( k \). Furthermore, there holds \( u = \bar{u}_{h_2} = \phi \) at the vertices of \( Q_1 \).

**Proof.** Due to Lemma 2.3 in [51] there exist polynomials \( \phi_1(x) \) and \( \phi_2(x) \), satisfying for integers \( \bar{k}_1, k_1 \) with \( 0 \leq \bar{k}_1 \leq k_1 \)
\[
\| \partial_x^{\alpha_1} (u(x, b_2) - \phi_1(x)) \|_{L^2(a_1, b_1)}^2 \leq C G(\alpha_1) \left( \| \partial_x^{k_1 + 1} u(x, b_2) \|_{L^2(a_1, b_1)}^2 \right) \cdot (7.31)
\]

and
\[
\left\| \partial_x^{\alpha_1} \left( \int_{a_2}^{b_2} \partial_z u(x, z) \, dz - \phi_2(x) \right) \right\|_{L^2(a_1, b_1)}^2 \leq C G(\alpha_1) \left( \| \partial_x^{k_1 + 1} \int_{a_2}^{b_2} \partial_z u(x, z) \, dz \right\|_{L^2(a_1, b_1)}^2 \cdot (7.32)
\]

where \( \alpha_1 = 0 \) or \( 1 \) with \( u(a_1, b_2) = \phi_1(a_1), u(b_1, b_2) = \phi_1(b_1) \) and \( \int_{a_2}^{b_2} \partial_z u(a_1, z) \, dz = \phi_2(a_1), \int_{a_2}^{b_2} \partial_z u(b_1, z) \, dz = \phi_2(b_1) \). Let
\[
\phi(x, y) := \phi_1(x) - \left( 1 - \frac{y - a_2}{h_2} \right) \phi_2(x). \cdot (7.33)
\]

In order to show (7.29) we proceed as follows.
\[
\| D^\alpha (\bar{u}_{h_2}(x, y) - \phi(x, y)) \|_{L^2(Q_1)}^2 \\
\leq 2 \| \partial_x^{\alpha_1} (u(x, b_2) - \phi_1(x)) \|_{L^2(Q_1)}^2 \\
+ 2 \left\| D^\alpha \left( \int_{a_2}^{b_2} \partial_z u(x, z) \, dz - \phi_2(x) \right) \right\|_{L^2(Q_1)}^2 \\
= 2 h_2 \| \partial_x^{\alpha_1} (u(x, b_2) - \phi_1(x)) \|_{L^2(a_1, b_1)}^2 \\
+ 2 \int_{a_2}^{b_2} \left( \frac{1}{h_2} - \frac{a_2}{h_2} \right)^2 \, dy \left\| \partial_x^{\alpha_1} \left( \int_{a_2}^{b_2} \partial_z u(x, z) \, dz - \phi_2(x) \right) \right\|_{L^2(a_1, b_1)}^2 \\
\leq 2 h_2 \| \partial_x^{\alpha_1} (u(x, b_2) - \phi_1(x)) \|_{L^2(a_1, b_1)}^2 \\
+ 2 h_2^{1-2\alpha_2} \left\| \partial_x^{\alpha_1} \left( \int_{a_2}^{b_2} \partial_z u(x, z) \, dz - \phi_2(x) \right) \right\|_{L^2(a_1, b_1)}^2 \\
\leq C h_2 G(\alpha_1) \left( \| \partial_x^{k_1 + 1} u(x, b_2) \|_{L^2(a_1, b_1)}^2 + C h_2^{1-2\alpha_2} G(\alpha_1) \right) \left\| \int_{a_2}^{b_2} \partial_z u(x, z) \, dz \right\|_{L^2(a_1, b_1)}^2 \cdot (7.34)
\]

From
\[
u(x, b_2) = u(x, b_2) - u(x, y) + u(x, y) = \int_{a_2}^{b_2} \partial_z u(x, z) \, dz + u(x, y)
\]
we obtain
\[
\frac{1}{h_2} \left\| \partial_x^{\tilde{k}_1+1} u(x, b_2) \right\|_{L^2(a_1, b_1)}^2 = \left\| \partial_x^{\tilde{k}_1+1} u(x, b_2) \right\|_{L^2(Q_1)}^2 \leq 2 \left\| \partial_x^{\tilde{k}_1+1} \int_y^{b_2} \partial_z u(x, z) dz \right\|_{L^2(Q_1)}^2 + 2 \left\| \partial_x^{\tilde{k}_1+1} u(x, y) \right\|_{L^2(Q_1)}^2. \quad (7.34)
\]

Altogether we have (7.29). To verify (7.30) we observe that due to (7.31), (7.33) and (7.34) there holds for any integer \( \tilde{k}_1, k_1 \) with \( 0 \leq \tilde{k}_1 \leq k_1 \)
\[
\| \partial_x^{\tilde{k}_1}(u(x, b_2) - \phi(x, b_2)) \|_{L^2(a_1, b_1)}^2 \leq C G(\alpha_1) \left\| \partial_x^{k_1+1} u(x, b_2) \right\|_{L^2(a_1, b_1)}^2 \leq C G(\alpha_1) \frac{1}{h_2} \left\{ \left\| \partial_x^{\tilde{k}_1+1} \int_y^{b_2} \partial_z u(x, z) dz \right\|_{L^2(Q_1)}^2 + \left\| \partial_x^{\tilde{k}_1+1} u(x, y) \right\|_{L^2(Q_1)}^2 \right\}. \quad (7.35)
\]

Due to (7.32) and (7.33) there holds for any integers \( \tilde{k}_1, k_1 \) with \( 0 \leq \tilde{k}_1 \leq k_1 \)
\[
\| \partial_x^{\tilde{k}_1}(u(x, a_2) - \phi(x, a_2)) \|_{L^2(a_1, b_1)}^2 \leq \| \partial_x^{\tilde{k}_1}(u(x, a_2) - \phi(x) + \phi(x)) \|_{L^2(a_1, b_1)}^2 = \| \partial_x^{\tilde{k}_1}(u(x, a_2) - u(x, b_2)) + \phi_2(x) + u(x, b_2) - \phi_1(x) \|_{L^2(a_1, b_1)}^2 \leq 2 \left\| \partial_x^{\tilde{k}_1} \int_a^{b_2} \partial_z u(x, z) dz - \phi_2(x) \right\|_{L^2(Q_1)}^2 + 2 \| \partial_x^{\tilde{k}_1}(u(x, b_2) - \phi_1(x)) \|_{L^2(a_1, b_1)}^2 \leq C G(\alpha_1) \frac{1}{h_2} \left\{ \left\| \partial_x^{\tilde{k}_1+1} u \right\|_{L^2(Q_1)}^2 + \left\| \partial_x^{\tilde{k}_1+1} \int_y^{b_2} \partial_z u(x, z) dz \right\|_{L^2(Q_1)}^2 + \left\| \partial_x^{\tilde{k}_1+1} \int_a^{b_2} \partial_z u(x, z) dz \right\|_{L^2(Q_1)}^2 \right\} \quad (7.36)
\]
for \( \alpha_1 = 0 \) or 1.

\[\square\]

Lemma 7.3 Let \( Q_1 = (a_1, b_1) \times (0, h_2) \subseteq Q = [0, 1]^2 \) with \( h_1 = b_1 - a_1 \leq \lambda_1 a_1, a_1 > 0, \lambda_1 \geq 0 \). For \( u \in H^{k+2,1}_\beta(Q) \) with weight function \( \Phi_{\beta, \alpha, 1}(x, y) \) in (7.22) there exist a polynomial \( \phi(x, y) = \sum_{0 \leq i \leq k_1} d_i x^i y^i \) on \( Q_1 \) with \( 1 \leq k_1 \leq k \) and a function
\[
\tilde{u}_{h_2}(x, y) := u(x, b_2) - (1 - \frac{y-a_2}{h_2}) \int_a^{b_2} \partial_z u(x, z) dz,
\]
such that for \( 0 \leq |\alpha| = \alpha_1 + \alpha_2 \leq 1 \) there holds with \( D^\alpha = \partial_x^{\alpha_1} \partial_y^{\alpha_2} \)
\[
\| D^\alpha(\tilde{u}_{h_2} - \phi) \|_{L^2(Q_1)}^2 \leq C a_1^{-2\alpha_1} \left( a_1^{2(1-\beta)} + h_2^{2(1-\beta-\alpha_2)} \right) \times \frac{\Gamma(k_1 - s_1 + 1)}{\Gamma(k_1 + s_1 + 3 - 2\alpha_1)} \left( \frac{\lambda_1}{2} \right)^{2(s_1+1-\alpha_1)} |u|_{H^{s_1+2,1}_\beta(Q)}^2. \quad (7.37)
\]
and for \( z = 0 \) or \( z = h_2 \)
\[
\| \partial_x^{\alpha_1}(\tilde{u}_{h_2}(x, z) - \phi(x, z)) \|_{L^2(a_1, b_1)}^2 \leq C h_{-1}^{-2\alpha_1} \left( h_2^{2(1-\beta)} + a_1^{2(1-\beta)} \right) \times \frac{\Gamma(k_1 - s_1 + 1)}{\Gamma(k_1 + s_1 + 3 - 2\alpha_1)} \left( \frac{\lambda_1}{2} \right)^{2(s_1+1-\alpha_1)} |u|_{H^{s_1+2,1}_\beta(Q)}^2. \quad (7.38)
\]
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Therefore, there holds the estimate

$$\left( \int_y^{y+h_2} \partial_z u(x,z) \, dz \right)^2 = \left( \int_y^{y+h_2} z^{-\beta} z^\beta \partial_z u(x,z) \, dz \right)^2 \leq \int_y^{y+h_2} z^{-2\beta} \left( z^\beta \partial_z u(x,z) \right)^2 \, dz$$

there holds the estimate

$$\left\| \partial_x^{\tilde{k}_1+1} \int_y^{y+h_2} \partial_z u(x,z) \, dz \right\|_{L^2(Q_1)}^2 \leq C h_2^{2-2\beta} \left\| \partial_x^{\tilde{k}_1+1} y^\beta \partial_y u(x,y) \right\|_{L^2(Q_1)}^2$$

and analogously

$$\left\| \partial_x^{\tilde{k}_1+1} \int_0^{h_2} \partial_z u(x,z) \, dz \right\|_{L^2(Q_1)}^2 \leq C h_2^{2-2\beta} \left\| \partial_x^{\tilde{k}_1+1} y^\beta \partial_y u(x,y) \right\|_{L^2(Q_1)}^2.$$ 

Then, due to Lemma 7.2, there exists a polynomial $\phi(x,y) = \sum_{0 \leq i \leq \tilde{k}_1, 0 \leq j \leq 1} d_{ij} x^i y^j$ on $Q_1$ with $1 \leq k_1 \leq k$, such that for $0 \leq \tilde{k}_1 \leq k_1$ there holds

$$\|D^\alpha (\tilde{u}_h (x,y) - \phi (x,y))\|_{L^2(Q_1)}^2 \leq C \left( \frac{h_1}{2} \right)^{2(\tilde{k}_1+1-\alpha_1)} \frac{(k_1-\tilde{k}_1)!}{(k_1 + \tilde{k}_1 + 2 - 2\alpha_1)!} \times \left\{ \left\| \partial_x^{\tilde{k}_1+1} u(x,y) \right\|_{L^2(Q_1)}^2 + h_2^{2(1-\beta-\alpha_2)} \left\| \partial_x^{\tilde{k}_1+1} y^\beta \partial_y u(x,y) \right\|_{L^2(Q_1)}^2 \right\}.$$ 

Due to Remark 7.2 the weight function $\Phi_{\beta,(\tilde{k}_1+1,1)}(x,y)$ becomes

$$\Phi_{\beta,(\tilde{k}_1+1,1)}(x,y) = x^{\tilde{k}_1+1} y + x^{\tilde{k}_1+1} y^\beta, \quad \tilde{k}_1 \geq 0. \quad (7.39)$$

Therefore

$$\left( \frac{h_1}{2} \right)^{2(\tilde{k}_1+1-\alpha_1)} \left\| \partial_x^{\tilde{k}_1+1} y^\beta \partial_y u(x,y) \right\|_{L^2(Q_1)}^2 \leq \left( \frac{\lambda_1}{2} \right)^{2(\tilde{k}_1+1-\alpha_1)} a_1^{2(\tilde{k}_1+1-\alpha_1)} a_1^{-2\tilde{k}_1-2} \left\| \partial_x^{\tilde{k}_1+1} y^\beta \partial_y u(x,y) \right\|_{L^2(Q_1)}^2$$

$$\leq \left( \frac{\lambda_1}{2} \right)^{2(\tilde{k}_1+1-\alpha_1)} a_1^{-2\alpha_1} \left\| \Phi_{\beta,(\tilde{k}_1+1,1)}(x,y) \partial_x^{\tilde{k}_1+1} \partial_y u(x,y) \right\|_{L^2(Q_1)}^2$$

$$\leq \left( \frac{\lambda_1}{2} \right)^{2(\tilde{k}_1+1-\alpha_1)} a_1^{-2\alpha_1} \left\| u \right\|_{H_\beta^{\tilde{k}_1+2,1}(Q)}^2. \quad (7.40)$$
and
\[
\left(\frac{h_1}{2}\right)^{2(k_1+1-\alpha)} \left\| \partial_x^{k_1+1} u(x,y) \right\|_{L^2(Q_1)}^2 \leq \left(\frac{\lambda_1}{2}\right)^{2(k_1+1-\alpha)} \left(\frac{\lambda_1}{2}\right)^{2(k_1+1-\alpha) - 2(\tilde{k}_1+\beta)} \left\| x^{\tilde{k}_1+\beta} \partial_x^{k_1+1} u(x,y) \right\|_{L^2(Q_1)}^2 \\
\leq \left(\frac{\lambda_1}{2}\right)^{2(k_1+1-\alpha)} \left(\frac{\lambda_1}{2}\right)^{2(k_1+1-\alpha) - 2(\tilde{k}_1+\beta)} \left\| \Phi_{\beta,\lambda_1}(x,y) \partial_x^{k_1+1} u(x,y) \right\|_{L^2(Q_1)}^2 \\
\leq \left(\frac{\lambda_1}{2}\right)^{2(k_1+1-\alpha)} \left(\frac{\lambda_1}{2}\right)^{2(k_1+1-\alpha) - 2(\tilde{k}_1+\beta)} \left\| u \right\|_{H^{k_1+1,1}(Q)}^2.
\]
(7.41)

Hence
\[
\left\| D^a(\tilde{u}_h(x,y) - \phi(x,y)) \right\|_{L^2(Q_1)}^2 \leq C \left(\frac{k_1 - \tilde{k}_1}{k_1 + k_1 + 2 - 2\alpha_1}\right)^{2(k_1+1-\alpha)} \left(\frac{\lambda_1}{2}\right)^{2(k_1+1-\alpha)} \\
\times \left( a_1^{-2\alpha_1} h_2^{2(1-\beta-\alpha_2)} \left\| u \right\|_{H^{k_1+2,1}(Q)}^2 + a_1^{-2(\alpha_1-\beta)} \left\| u \right\|_{H^{k_1+1,1}(Q)}^2 \right) \\
\leq C \left( a_1^{-2\alpha_1} h_2^{2(1-\beta-\alpha_2)} \right) \left(\frac{k_1 - \tilde{k}_1}{k_1 + k_1 + 2 - 2\alpha_1}\right)^{2(k_1+1-\alpha)} \left(\frac{\lambda_1}{2}\right)^{2(k_1+1-\alpha)} \left\| u \right\|_{H^{k_1+1,1}(Q)}^2.
\]
(7.42)

Due to Lemma 7.2 there holds for any integer \( \tilde{k}_1, k_1 \) with \( 0 \leq \tilde{k}_1 \leq k_1 \) together with (7.40), (7.41) for \( z = 0 \) or \( z = h_2 \)
\[
\left\| \partial_x^{\alpha_1} (u(x,z) - \phi(x,z)) \right\|_{L^2(a_1,b_1)}^2 \leq C \left(\frac{h_1}{2}\right)^{2(k_1+1-\alpha)} \left(\frac{k_1 - \tilde{k}_1}{k_1 + k_1 + 2 - 2\alpha_1}\right)^{2(k_1+1-\alpha)} \\
\times \left( a_1^{-2\alpha_1} h_2^{2(1-\beta-\alpha_2)} \left\| x^{\tilde{k}_1+1} y^\beta \partial_y u(x,y) \right\|_{L^2(Q_1)}^2 + \left\| \partial_x^{k_1+1} u(x,y) \right\|_{L^2(Q_1)}^2 \right) \\
\leq C \left(\frac{k_1 - \tilde{k}_1}{k_1 + k_1 + 2 - 2\alpha_1}\right)^{2(k_1+1-\alpha)} \left(\frac{\lambda_1}{2}\right)^{2(k_1+1-\alpha)} \left\| u \right\|_{H^{k_1+1,1}(Q)}^2.
\]
(7.43)

Let \( T_a u = D^a(\tilde{u}_h - \phi) \). Then \( T_a \) is a linear operator from \( H^{\tilde{k}_1+2,1}(Q) \) into \( L^2(Q_1) \) for any integers \( \tilde{k}_1, k_1 \) with \( 0 \leq \tilde{k}_1 \leq k_1 \), and the norm of the operator is bounded as
\[
\left\| T_a \right\|_{H^{\tilde{k}_1+2,1}(Q),L^2(Q_1)}^2 \leq C \left( a_1^{-2\alpha_1} h_2^{2(1-\beta-\alpha_2)} \right) \left(\frac{k_1 - \tilde{k}_1}{k_1 + k_1 + 2 - 2\alpha_1}\right)^{2(k_1+1-\alpha)} \left(\frac{\lambda_1}{2}\right)^{2(k_1+1-\alpha)}.
\]
Standard interpolation arguments show that $T_\alpha$ is also a bounded linear operator from $H^{s_1+2, 1}_\beta(Q)$ into $L^2(Q_1)$ for real $s_1$ with $0 \leq s_1 \leq k_1$, and there holds

$$\|T_\alpha\|_{H^{s_1+2, 1}_\beta(Q), L^2(Q_1)} \leq C \epsilon^{2\alpha_1} (a_1^{2(1-\beta)} + b_2^{2(1-\beta-\alpha_2)}) \left( \frac{\Lambda_1}{2} \right)^{2(s_1+1-\alpha_1)} \frac{\Gamma(k_1-s_1+1)}{\Gamma(k_1+s_1+3-2\alpha_1)}.$$  

This completes the proof of Lemma 7.3 $\square$

**Lemma 7.4** (c.f. [67, Lemma 3.8]) Let $S = (-1, 1) \times (-1, 1)$ and $u \in H^{k+3}(S)$ with $k = \max(k_1, k_2)$, $k_1, k_2 \geq 2$. Then, there exist $u_1, u_2 \in H^{k+3}(S)$ with $u = u_1 + u_2$ and polynomials $\phi_i(x, y) = \sum_{0 \leq i \leq k_1, 0 \leq j \leq k_2} d_{i,j}^{(l)} x^i y^j$, ($l = 1, 2$), such that for all integers $1 \leq s_i \leq k_i$, $i = 1, 2$, and $0 \leq |\alpha| \leq 2$ there holds

$$\|D^\alpha(u_1 - \phi_1)\|_{L^2(S)}^2 \leq C \frac{(k_1-s_1)!}{(k_1+s_1+2-2\alpha_1)!} \sum_{l=\alpha_1}^2 \left\| \frac{\partial^{s_1+1+l} u}{\partial x^{s_1+1} \partial y^l} \right\|_{L^2(S)}^2,$$

$$\|D^\alpha(u_2 - \phi_2)\|_{L^2(S)}^2 \leq C \frac{(k_2-s_2)!}{(k_2+s_2+2-2\alpha_2)!} \sum_{l=\alpha_1}^2 \left\| \frac{\partial^{s_2+1+l} u}{\partial x^l \partial y^{s_2+1}} \right\|_{L^2(S)}^2 \quad (7.44)$$

with $C$ not depending on $k_i$. There holds also $u = \phi_1 + \phi_2$ in the corners of $S$.

**Proof.** Let $p_i(x)$ (resp. $p_j(y)$) be the Legendre polynomials on $(-1, 1)$ of degree $i$ (resp. $j$). Due to the assumptions there holds $u \in H^5(S)$. Therefore there hold the Fourier expansions

$$\partial_x^2 \partial_y^2 u(x, y) = \sum_{i,j=0}^\infty c_{i,j} p_i(x) p_j(y),$$

$$\partial_y^2 \partial_x^2 u(x, y) \big|_{y=-1} = \sum_{i=0}^\infty a_i p_i(x), \quad \partial_x^2 \partial_y^2 u(x, y) \big|_{x=-1} = \sum_{j=0}^\infty b_j p_j(y),$$

$$\partial_x^2 \partial_y^2 u(x, y) \big|_{y=1} = \sum_{i=0}^\infty d_{i} p_i(x), \quad \partial_x^2 \partial_y^2 u(x, y) \big|_{x=1} = \sum_{j=0}^\infty e_j p_j(y).$$

Due to [46, Lemma 4.2] there exists functions $u_1, u_2 \in H^{k+3}(S)$ with $u = u_1 + u_2$ and polynomials $\phi_i(x, y) = \sum_{0 \leq i \leq k_1, 0 \leq j \leq k_2} d_{i,j}^{(l)} x^i y^j$, ($l = 1, 2$) with $\phi = \phi_1 + \phi_2$ such that
u_1 - \phi_1 = F_1 + F_2 + F_3 \quad \text{and} \quad u_2 - \phi_2 = F_4 + F_5 + F_6 \quad \text{where}

\begin{align*}
F_1 &= \int \int \int \int_{-1}^{x} y \int_{-1}^{y} \xi_1 \xi_2 \sum_{i \geq k_1 - 1} c_{i,j} p_i(\xi_1) p_j(\xi_2) d\xi_1 d\xi_2 d\xi_1 d\xi_2, \\
F_4 &= (1 + y) \int \int \int_{-1}^{x} y \int_{-1}^{y} \xi_1 \sum_{i \geq k_1 - 1} a_i p_i(\xi_1) d\xi_1, \\
F_5 &= (1 + x) \int \int \int_{-1}^{y} x \int_{-1}^{y} \xi_2 \sum_{j \geq k_2 - 1} b_j p_j(\xi_2) d\xi_2, \\
F_3 &= \int \int \int_{-1}^{x} \xi_1 \sum_{i \geq k_1 - 1} d_i p_i(\xi_1) d\xi_1 d\xi_1, \\
F_6 &= \int \int \int_{-1}^{y} \xi_2 \sum_{j \geq k_2 - 1} e_j p_j(\xi_2) d\xi_2 d\xi_2.
\end{align*}

Lemma 4.2 in [46] gives an estimate of the approximation error in terms of the total derivative \( D^m = \sum_{m=|\alpha|} D^\alpha \). Here, we need estimates of \( D^\alpha (u_1 - \phi_1) \). Therefore, we have to investigate the derivatives \( D^\alpha F_j \) \((j = 1, \ldots, 6)\) individually.

One obtains

\begin{align*}
\frac{\partial^2 F_1}{\partial x^2} &= \int \int \xi_2 \sum_{i \geq k_1 - 1} \sum_{j \geq 0} c_{i,j} p_i(x) p_j(\xi_2) d\xi_2 d\xi_2 \\
&= \sum_{i \geq k_1 - 1} c_{i,j} p_i(x) \frac{1}{2j + 1} \left( \frac{p_{j+2}(y) - p_j(y)}{2j + 3} - \frac{p_j(y) - p_{j-2}(y)}{2j - 1} \right)
\end{align*}

and

\begin{equation*}
\left\| \frac{\partial^2 F_1}{\partial x^2} \right\|_{L^2(S)}^2 \leq C \sum_{i \geq k_1 - 1} \sum_{j \geq 0} |c_{i,j}|^2 \frac{2}{2j + 1} \frac{2}{12j + 1} \frac{2}{(2j + 1)^4}.
\end{equation*}
For $0 \leq |\alpha| \leq 2$ and integers $1 \leq s_i \leq k_i$ ($i = 1, 2$) one gets with Lemma 4.1 in [46]
\[
\|D^\alpha F_1\|_{L^2(\mathcal{S})}^2 \leq \sum_{i \geq k_1-1} |c_{i,j}|^2 \frac{2}{2i+1} \frac{2}{2j+1} \frac{2}{(2i+1)(2j+1)(2i+2)} \sum_{i \geq s_1-1} |c_{i,j}|^2 \frac{2}{2i+1} \frac{2}{2j+1} \frac{2}{(i+s_1-1)!} \frac{2}{(i-(s_1-1))!}
\]
where $C$ is independent of $k$. Correspondingly, we have
\[
\|D^\alpha F_4\|_{L^2(\mathcal{S})}^2 \leq \sum_{i \geq k_1-1} \frac{(k_2-s_2)!}{(k_2+s_2+2-2\alpha_2)!} \frac{2}{(k_2+s_2+2-2\alpha_2)!} \frac{2}{(i+s_1-1)!} \frac{2}{(i-(s_1-1))!} \frac{2}{(i-(s_1-1))!}
\]
Next, we estimate $F_2$. Using $\frac{\partial^2 F_2}{\partial x^2} = (1+y) \sum_{i \geq k_1-1} a_i p_i(x)$ we find
\[
\|\frac{\partial^2 F_2}{\partial x^2}\|_{L^2(\mathcal{S})}^2 \leq C \sum_{i \geq k_1-1} \frac{a_i^2}{2i+1} \leq C \frac{(k_1-s_1)!}{(k_1+s_1-2)!} \sum_{i \geq s_1-1} \frac{2}{2i+1} \frac{2}{(i+s_1-1)!} \frac{2}{(i-(s_1-1))!}
\]
Correspondingly, we obtain
\[
\|\frac{\partial^2 F_2}{\partial x^{\alpha_1}}\|_{L^2(\mathcal{S})}^2 \leq C \frac{(k_1-s_1)!}{(k_1+s_1+2-2\alpha_1)!} \frac{2}{(k_1+s_1+2-2\alpha_1)!} \frac{2}{(i+s_1-1)!} \frac{2}{(i-(s_1-1))!}
\]
Obviously $\frac{\partial^2 F_2}{\partial y^2} = 0$ and $\|\frac{\partial F_2}{\partial y}\|_{L^2(\mathcal{S})} \leq C \|F_2\|_{L^2(\mathcal{S})}$. Hence, for $0 \leq |\alpha| \leq 2$ there holds
\[
\|D^\alpha F_2\|_{L^2(\mathcal{S})}^2 \leq C \frac{(k_1-s_1)!}{(k_1+s_1+2-2\alpha_1)!} \frac{2}{(k_1+s_1+2-2\alpha_1)!} \frac{2}{(i+s_1-1)!} \frac{2}{(i-(s_1-1))!}
\]
and similarly
\[
\|D^\alpha F_5\|_{L^2(\mathcal{S})}^2 \leq C \frac{(k_2-s_2)!}{(k_2+s_2+2-2\alpha_2)!} \frac{2}{(k_2+s_2+2-2\alpha_2)!} \frac{2}{(i+s_1-1)!} \frac{2}{(i-(s_1-1))!}
\]
Furthermore, $\frac{\partial F_2}{\partial y}$, $\frac{\partial^2 F_3}{\partial y^2}$ and $\frac{\partial^2 F_3}{\partial y^2}$ vanish and for $0 \leq \alpha_1 \leq 2$, $\alpha_2 = 0$
\[
\|D^\alpha F_3\|_{L^2(\mathcal{S})}^2 \leq C \frac{(k_1-s_1)!}{(k_1+s_1+2-2\alpha_1)!} \frac{2}{(k_1+s_1+2-2\alpha_1)!} \frac{2}{(i+s_1-1)!} \frac{2}{(i-(s_1-1))!}
\]
similarly for $0 \leq \alpha_2 \leq 2$, $\alpha_1 = 0$

$$\|D^a F_6\|^2_{L^2(S)} \leq C \frac{(k_2 - s_2)!}{(k_2 + s_2 + 2 - 2a_2)!} \left\| \frac{\partial^a_2 + 1 u}{\partial y^{s_2 + 1}} \right\|^2_{L^2(S)}.$$  \hfill (7.50)

Combining the above estimates we find the assertion (7.44). The orthogonality of the Legendre polynomials implies $F_1(\pm 1, y) = F_2(x, \pm 1) = F_3(\pm 1) = F_6(\pm 1) = 0$. Hence the difference of $u$ and $\phi$ vanishes at the vertices of $S$. \hfill $\boxdot$

**Lemma 7.5** Let $Q_1 = (a_1, b_1) \times (a_2, b_2)$ with $h_1 = b_1 - a_1$ and $h_2 = b_2 - a_2$ and $u \in H^{k+3}(Q_1)$, $k = \max(k_1, k_2)$, $k_1, k_2 \geq 2$. Then there exist $u_1, u_2 \in H^{k+3}(Q_1)$ with $u = u_1 + u_2$ and polynomials $\phi_l(x, y) = \sum_{0 \leq l \leq k_1, 0 \leq j \leq k_2} d_{i,j}^l x^i y^j$, $(l = 1, 2)$, such that for all integers $1 \leq s_i \leq k_i$, $i = 1, 2$, and $0 \leq |\alpha| \leq 2$ there holds

$$\|D^\alpha (u_1 - \phi_1)\|^2_{L^2(Q_1)} \leq C \frac{(k_1 - s_1)!}{(k_1 + s_1 + 2 - 2\alpha_1)!} \left( \frac{h_1}{2} \right)^{2(\alpha_1 + 1 - \alpha_2)} \sum_{l=2}^{\alpha_2} \left\| \frac{\partial^l x_{s_1 + 1} u}{\partial x^l \partial y} \right\|^2_{L^2(Q_1)} \left( \frac{h_2}{2} \right)^{2(l-\alpha_2)}.$$  \hfill (7.51)

$$\|D^\alpha (u_2 - \phi_2)\|^2_{L^2(Q_1)} \leq C \frac{(k_2 - s_2)!}{(k_2 + s_2 + 2 - 2\alpha_2)!} \left( \frac{h_2}{2} \right)^{2(\alpha_2 + 1 - \alpha_1)} \sum_{l=2}^{\alpha_1} \left\| \frac{\partial^l x_{s_2 + 1} u}{\partial x^l \partial y^2} \right\|^2_{L^2(Q_1)} \left( \frac{h_1}{2} \right)^{2(l-\alpha_1)}$$

with $C$ independent of $k_i$. There also holds $u = \phi_1 + \phi_2$ in the corners of $Q_1$. \hfill $\boxdot$

**Proof.** By applying a scaling argument to Lemma 7.4. \hfill $\boxdot$

**Remark 7.1** Lemma 7.5 corresponds in crucial parts to a result by Guo and Babuška [46, Lemma 4.2]. The difference is in the interpretation of the approximation result. In [46, Lemma 4.2] there are additional terms involved. By removing those terms in Lemma 7.4 we can prove convergence of the hp version BEM in the $B^{1}_{\infty}$-space from Def. 7.4. Consequently we use $D^a u$ instead of the total derivative $|D^a u|^2 := \sum_{|\alpha|=n} |D^a u|^2$.

**Theorem 7.6** Let $Q_1 = (a_1, b_1) \times (a_2, b_2) \subset \subset Q$ with $h_1 = b_1 - a_1 \leq \lambda_1 a_1$, $h_2 = b_2 - a_2 \leq \lambda_2 a_2$, $\lambda_i \geq 0$, $(i = 1, 2)$, and $u \in H^{k+3}_{\beta} (Q)$ with $\Phi_{\beta, \alpha, 1} (x, y)$ used in (7.15). Then, there exists a polynomial $\phi(x) = \sum_{0 \leq i,j \leq k_1, 0 \leq j \leq k_2} d_{i,j} x^i y^j$ on $Q_1$ with $2 \leq k_1, k_2 \leq k$, such that for $0 \leq \alpha_1, \alpha_2 \leq 1$ there holds

$$\|D^a (u - \phi)\|^2_{L^2(Q_1)} \leq C \sum_{j=1}^2 a_j \left( \frac{\Gamma(k_j - s_j + 1) \Gamma(k_j + s_j + 3 - 2\alpha_j)}{\Gamma(k_j + s_j + 2 - 2\alpha_j)} \right)^{2\alpha_j} \left| u \right|_{H^{s_j + 3.1}_{\beta} (Q)}^2.$$  \hfill (7.52)

where $s_j$ is a real number $1 \leq s_j \leq k_j$ $(j = 1, 2)$ and $H^{s_j + 3.2}_{\beta} (Q)$ is the interpolation space $(H^{k_j + 2.1}_{\beta} (Q), H^{k_j + 3.1}_{\beta} (Q))_{\theta,\infty}$ for integers $k_j = s_j + 1 - \theta_j \leq k_j$, $0 \leq \theta_j \leq 1$ and $C$ is independent of $k$, but depending on $\lambda_j$ $(j = 1, 2)$. There also holds $u = \phi$ in the corners of $Q_1$. \hfill $\boxdot$
Proof. Due to Lemma 7.5 we have the decomposition \( u - \phi = u_1 - \phi_1 + u_2 - \phi_2 \). The individual terms can be bound for integers \( \bar{k}_j, (1 \leq \bar{k}_j \leq k_j, j = 1, 2) \) in the following way

\[
\|D^\alpha (u_1 - \phi_1)\|_{L^2(Q)}^2 \leq C \frac{(k_1 - \bar{k}_1)!}{(k_1 + \bar{k}_1 + 2 - 2\alpha_1)!} \left( \frac{\lambda_1 \alpha_1}{2} \right)^{2(k_1 + 1 - \alpha_1)} \sum_{l=0}^{2} \left\| \frac{\partial^{k_1+1+l} u}{\partial x^{k_1+1+l} y^l} \right\|_{L^2(Q)}^2 \left( \frac{\lambda_2 \alpha_2}{2} \right)^{2(l - \alpha_2)}
\]

Due to Remark 7.2, we have for \( x, y \geq 0 \) with \( \bar{k}_1 \geq 1 \)

\[
x^{\beta + \bar{k}_1 + 1 - 1} y^0 = \Phi_{\beta, (\bar{k}_1 + 1, 0), 1},
\]

\[
x^{\beta + \bar{k}_1 + 1 - 1} y^1 \leq \Phi_{\beta, (\bar{k}_1 + 1, 1), 1} = x^{\beta + \bar{k}_1 + 1 - 1} y^1 + x^{\bar{k}_1 + 1} y^{\beta + 1 - 1},
\]

\[
x^{\beta + \bar{k}_1 + 1 - 1} y^2 \leq \Phi_{\beta, (\bar{k}_1 + 1, 2), 1} = x^{\beta + \bar{k}_1 + 1 - 1} y^2 + x^{\bar{k}_1 + 1} y^{\beta + 2 - 1}.
\]

Therefore, we find

\[
\|D^\alpha (u_1 - \phi_1)\|_{L^2(Q)}^2 \leq C a_1^{2(1 - \beta - \alpha_1)} a_2^{-2\alpha_2} \frac{(k_1 - \bar{k}_1)!}{(k_1 + \bar{k}_1 + 2 - 2\alpha_1)!} \left( \frac{\lambda_1}{2} \right)^{2(\bar{k}_1 + 1 - \alpha_1)} \sum_{l=0}^{2} \left\| \Phi_{\beta, (\bar{k}_1 + 1, l), 1} \frac{\partial^{k_1+1+l} u}{\partial x^{k_1+1+l} y^l} \right\|_{L^2(Q)}^2 \left( \frac{\lambda_2}{2} \right)^{2(l - \alpha_2)}
\]

Analogously we have

\[
\|D^\alpha (u_2 - \phi_2)\|_{L^2(Q)}^2 \leq C a_1^{2(1 - \beta - \alpha_1)} a_2^{-2\alpha_1} \frac{(k_2 - \bar{k}_2)!}{(k_2 + \bar{k}_2 + 2 - 2\alpha_1)!} \left( \frac{\lambda_1}{2} \right)^{2(k_2 + 1 - \alpha_2)} \left| u \right|_{H^{\bar{k}_1+3,1}_\beta (Q)}^2 \max \left\{ \left( \frac{\lambda_2}{2} \right)^{2(\bar{k}_1 + 1 - \alpha_1)}, 1 \right\}
\]

Let \( T_{j,\alpha} u = D^\alpha (u_j - \phi_j) \ (j = 1, 2), (0 \leq \alpha_1, \alpha_2 \leq 1) \). Then \( T_{j,\alpha} \) is a linear operator from \( H^{\bar{k}_j+3,1}_\beta (Q) \) into \( L^2(Q) \) for all integers \( 1 \leq \bar{k}_j \leq k_j \) and the norm of the operator is bounded

\[
\|T_{j,\alpha}\|_{H^{\bar{k}_j+3,1}_\beta (Q), L^2(Q)}^2 \leq C a_j^{2(1 - \alpha_j - \beta)} a_{3-j}^{-2\alpha_{3-j}} \frac{(k_j - \bar{k}_j)!}{(k_j + \bar{k}_j + 2 - 2\alpha_j)!} \left( \frac{\lambda_j}{2} \right)^{2(\bar{k}_j + 1 - \alpha_j)} \Gamma(k_j - s_j + 1)
\]

Standard interpolation arguments show that \( T_{j,\alpha} \) is also a linear and continuous operator from \( H^{s_j+3,1}_\beta (Q) \) into \( L^2(Q) \) for real \( s_j \) with \( 1 \leq s_j \leq k_j \), and there holds for \( j = 1, 2 \)

\[
\|T_{j,\alpha}\|_{H^{s_j+3,1}_\beta (Q), L^2(Q)}^2 \leq C a_j^{2(1 - \alpha_j - \beta)} a_{3-j}^{-2\alpha_{3-j}} \left( \frac{\lambda_j}{2} \right)^{2(s_j + 1 - \alpha_j)} \frac{\Gamma(k_j - s_j + 1)}{\Gamma(k_j + s_j + 3 - 2\alpha_j)}
\]
and therefore
\[ \|D^a (u - \phi)\|_{L^2(Q_1)}^2 \leq C \sum_{j=1}^{2} a_j^2 (1 - \alpha_j - \beta) a_{3-j}^{-2\alpha_{3-j}} \frac{\Gamma(k_j - s_j + 1)}{\Gamma(k_j + s_j + 3 - 2\alpha_j)} \left( \frac{\lambda_j}{2} \right)^{2(s_j + 1 - \alpha_j)} |u|_{H_{\beta_j+3,1}^1(Q)}^2. \]

This completes the proof of Theorem 7.6. \(\square\)

**Lemma 7.6** Let \( \gamma_i \) be the sides of the rectangle \( T = [0, h_1] \times [0, h_2], 1 \leq i \leq 4 \). Assume that \( \gamma_1 \) lies on the \( x \)-axis and that \( v(x) \) is a polynomial of degree \( p \) on \( \gamma_1 \) and that \( v \) vanishes at the endpoints of \( \gamma_1 \). Then, the polynomial \( V(x, y) = v(x)(1 - y/h_2) \) of degree \( p \) in \( x \) and degree 1 in \( y \) satisfies \( V = 0 \) on \( \gamma_1 \) and \( V = v \) on \( \gamma_1 \), and
\[ \|\partial_z^{\alpha_1} \partial_y^{\alpha_2} V \|_{L^2(T)}^2 \leq h_1^{2-2\alpha_1} h_2^{1-2\alpha_2} \|\partial_x v\|_{L^2(\gamma_1)}^2, \quad 0 \leq \alpha_1, \alpha_2 \leq 1. \] (7.53)

**Proof.** Obviously, we have \( V(x, h_2) = V(0, y) = V(h_1, y) = 0 \) and \( V(x, 0) = v(x) \). \( V(x, y) \) is a polynomial of degree \( p \) in \( x \) and degree 1 in \( y \), and
\[ \|\partial_z^{\alpha_1} \partial_y^{\alpha_2} V(x, y)\|_{L^2(T)}^2 = \int_0^{h_2} (\partial_y^{\alpha_2}(1 - y/h_2))^2 dy \int_0^{h_1} (\partial_z^{\alpha_1} v(x))^2 dx. \] (7.54)

Due to \( v(0) = v(h_1) = 0 \) there holds
\[ v^2(x) = \left( v(0) + \int_0^x \partial_z v(z) dz \right)^2 = \left( \int_0^x \partial_z v(z) dz \right)^2 \leq x \int_0^x (\partial_z v(z))^2 dz \]
and consequently
\[ \int_0^{h_1} v^2(x) dx \leq \int_0^{h_1} x \int_0^x (\partial_z v(z))^2 dz dx \leq \frac{h_1^2}{2} \int_0^{h_1} (\partial_z v(z))^2 dz. \]

Therefore, we obtain
\[ \int_0^{h_1} (\partial_z^{\alpha_1} v(x))^2 dx \leq h_1^{2-2\alpha_1} \int_0^{h_1} (\partial_x v(x))^2 dx, \quad 0 \leq \alpha_1 \leq 1. \] (7.55)

Also there holds
\[ \int_0^{h_2} (\partial_y^{\alpha_2}(1 - y/h_2))^2 dy = (1/3)^{1-\alpha_2} h_2^{1-2\alpha_2}, \quad 0 \leq \alpha_2 \leq 1. \] (7.56)

Therefore, we obtain (7.53) for the rectangle \( T \). \(\square\)

**Lemma 7.7** Let \( T = [0, h_1] \times [0, h_2] \) and \( u \in H^2(T) \). Then, we have
\[ \int_0^{h_1} (\partial_x u(x, 0))^2 dx \leq \frac{2}{h_2} \|\partial_x u\|_{L^2(T)}^2 + h_2 \|\partial_x \partial_y u\|_{L^2(T)}^2. \] (7.57)

**Proof.** We have
\[
\int_0^{h_1} \int_0^{h_2} (\partial_x u(x, y))^2 dy dx = \int_0^{h_1} \int_0^{h_2} (\partial_x u(x, y) - \int_0^y \partial_x \partial_z u(x, z) dz)^2 dy dx \\
\leq 2 \int_0^{h_1} \int_0^{h_2} (\partial_x u(x, y))^2 dy dx + 2 \int_0^{h_1} \int_0^{h_2} (\partial_x \partial_z u(x, y))^2 dy dx \\
\leq 2 \|\partial_x u\|_{L^2(T)}^2 + 2 \int_0^{h_1} \int_0^{h_2} (\partial_x \partial_y u(x, z))^2 dz dy dx \\
\leq 2 \|\partial_x u\|_{L^2(T)}^2 + h_2^3 \|\partial_x \partial_y u\|_{L^2(T)}^2.
\]

Therefore, we obtain (7.57) for the rectangle \( T \). \(\square\)
Lemma 7.8 Let $0 \leq \beta < 1/2$, $h > 0$ and $u$ sufficiently differentiable, then there holds

$$
\int_0^h x \left( \partial_x u(x) - \frac{1}{h} \int_0^h \partial_z u(z) \, dz \right)^2 \, dx \leq C h^{1-2\beta} \int_0^h (x^{1+\beta} \partial_x^2 u(x))^2 \, dx
$$

(7.58)

with $C$ independent on $h$ and $u$, but dependent on $\beta$.

Proof. First, we have

$$
\int_0^h x \left( \partial_x u(x) - \frac{1}{h} \int_0^h \partial_z u(z) \, dz \right)^2 \, dx = \frac{1}{h^2} \int_0^h x \left( \int_0^h \int_z^x \partial^2_w u(w) \, dw \, dz \right)^2 \, dx.
$$

Let $\varepsilon > 0$ such that $\beta + \varepsilon < 1/2$. Then, we have

$$
\left( \int_0^h \int_z^{\varepsilon} \partial^2_w u(w) \, dw \, dz \right)^2 \leq \frac{h^{1-2\beta-2\varepsilon}}{1-2\beta-2\varepsilon} \int_0^h \int_z^{2\varepsilon} \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dz.
$$

Now, we have to split the inner integral to proceed with our bound.

$$
\int_0^h \int_z^{2\beta+2\varepsilon} \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dz
= \int_0^h \int_z^{2\beta+2\varepsilon} \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dz + \int_0^h \int_z^{2\beta+2\varepsilon} \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dz.
$$

For the first integral we obtain the following upper bound

$$
\int_0^h \int_z^{2\beta+2\varepsilon} \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dz
\leq \int_0^h \int_z^{2\beta+2\varepsilon} \int_z^{x} \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dx \, dz
\leq \int_0^h \int_z^{2\beta+2\varepsilon} \frac{x^{2\varepsilon-2\beta-1} - z^{2\beta-1}}{1+2\beta} \, dz \int_0^x \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dx
= \frac{1}{1+2\beta} \left( \frac{x^{2\varepsilon}}{2\varepsilon} - \frac{x^{2\beta+2\varepsilon}}{2\beta+2\varepsilon+1} \right) \int_0^x \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dx
= \frac{x^{2\varepsilon}}{2\varepsilon(2\beta+2\varepsilon+1)} \int_0^x \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dx.
$$

Analogously, we obtain for the second integral

$$
\int_0^h \int_z^{2\beta+2\varepsilon} \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dz
\leq \int_0^h \int_z^{2\beta+2\varepsilon} \int_z^{x} \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dx \, dz
\leq \int_0^h \int_z^{2\beta+2\varepsilon} \frac{x^{2\beta-1} - z^{2\beta-1}}{1+2\beta} \, dz \int_0^x \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dx
= \frac{1}{1+2\beta} \left( \frac{x^{2\beta-1} h^{2\beta+2\varepsilon+1} - x^{2\beta+2\varepsilon+1}}{2\beta+2\varepsilon+1} - \frac{h^{2\varepsilon}}{2\varepsilon} \right) \int_0^x \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dx
= \left( \frac{x^{2\varepsilon}}{2\varepsilon(2\beta+2\varepsilon+1)} + \frac{1}{1+2\beta} \left( x^{2\beta-1} - \frac{h^{2\beta+2\varepsilon+1}}{2\beta+2\varepsilon+1} \right) \right) \int_0^x \left( \int_z^x \partial^2_w u(w) \, dw \right)^2 \, dx.
$$
Fitting both parts together, we obtain

\[
\int_0^h z^{2\beta+2\varepsilon} \left( \int_x^z \frac{\partial^2_{ul} u(w)}{z} dw \right)^2 dz \\
\leq \left( \frac{1}{2\varepsilon(2\beta + 2\varepsilon + 1)} + \frac{1}{1 + 2\beta} \left( x^{-2\beta-1} \frac{h^{2\beta + 2\varepsilon + 1}}{2\beta + 2\varepsilon + 1} - \frac{h\varepsilon}{2\varepsilon} \right) \right) \int_0^h \left( w^{1+\beta \partial^2_{ul} u(w)} \right)^2 dw
\]

and, consequently

\[
\int_0^h x \left( \partial_x u(x) - \frac{1}{h} \int_0^h \partial_z u(z) dz \right)^2 dx \\
\leq \frac{1}{h^2} \left( \frac{h^{1-2\beta-2\varepsilon}}{1 - 2\beta - 2\varepsilon} \left( \frac{h^{\varepsilon+2/(2\varepsilon + 2)}}{2\varepsilon(2\beta + 2\varepsilon + 1)} + \frac{1}{1 + 2\beta} \left( \frac{h^{1-2\beta} h^{2\varepsilon+1}}{1 - 2\beta 2\beta + 2\varepsilon + 1} - \frac{h\varepsilon^2/2}{2\varepsilon} \right) \right) \right) \int_0^h \left( w^{1+\beta \partial^2_{ul} u(w)} \right)^2 dw
\]

Let \( I_x \) be the linear interpolation operator with respect to the variable \( x \) on the interval \([0, h]\), e.g., \( I_x u(x) := \frac{x}{h} u(h) + \frac{h-x}{h^2} u(0) = u(h) - (1 - \frac{x}{h}) \int_0^h \partial_z u(z) dz \).

**Lemma 7.9** Let \( 0 \leq \beta < 1/2, h > 0 \) and \( u \) sufficiently differentiable, then there holds

\[
\int_0^h x (u(x) - I_x u(x))^2 dx \leq 2 \frac{h^{3-2\beta}}{1 - 2\beta} \int_0^h (x^2 \partial_x u(x))^2 dx. \tag{7.59}
\]

**Proof.** Firstly, we have

\[
\int_0^h x (u(x) - I_x u(x))^2 dx = \int_0^h x \left( u(x) - \left( u(h) - (1 - \frac{x}{h}) \int_0^h \partial_z u(z) dz \right) \right)^2 dx \\
\leq 2 \int_0^h x \left( \int_x^h \partial_z u(z) dz \right)^2 dx + 2 \int_0^h x \left( \frac{h-x}{h} \int_0^h \partial_z u(z) dz \right)^2 dx.
\]

The first term can be bound by

\[
\int_0^h x \left( \int_x^h \partial_z u(z) dz \right)^2 dx \leq \int_0^h x \int_x^h z^{-2\beta} dz \int_x^h \left( z^\beta \partial_z u(z) \right)^2 dz dx \\
\leq \int_0^h x \frac{h^{1-2\beta} - x^{1-2\beta}}{1 - 2\beta} dx \int_0^h \left( z^\beta \partial_z u(z) \right)^2 dz \\
= \frac{h^{3-2\beta}}{2(3-2\beta)} \int_0^h \left( z^\beta \partial_z u(z) \right)^2 dz.
\]
The second term is bounded by
\[
\int_0^h x \left( \frac{h-x}{h} \int_0^h \partial_x u(z) \, dz \right)^2 \, dx \leq \frac{h}{12} \int_0^h \left( z^\beta \partial_x u(z) \right)^2 \, dz.
\]
\[\square\]

**Lemma 7.10** Let \( Q = [0, 1]^2 \). There holds
\[
\| v \|^2_{H^1(Q)} \leq C \{ \| y^{1/2} v(x, y) \|^2_{L^2(Q)} + \| y^{1/2} \partial_x v(x, y) \|^2_{L^2(Q)} \}.
\]

**Proof.** Let \( S := \{(x, y, z) : \sqrt{y^2 + z^2} < 1, (x, y, z) \in (0, 1)^3\} \). Then we can define \( V |_S \) in cylindrical coordinates \((\rho, \varphi, x)\) with \( \rho = \sqrt{y^2 + z^2} \) by \( V(\rho, \varphi, x) := v(x, y) \), i.e., we have defined \( V |_S \) by rotating \( v(x, y) \), \((x, y) \in Q\) around the \( x \)-axis. The \( H^1(S) \) norm of \( V \) is given by
\[
\| V \|^2_{H^1(S)} = \| V \|^2_{L^2(S)} + \| \partial_x V \|^2_{L^2(S)} + \frac{1}{2} \| \partial_x \rho V \|^2_{L^2(S)} + \| \partial_x V \|^2_{S(S)}
\]
Due to the trace theorem (see, e.g., [1]), we have
\[
\| v \|^2_{H^1(\partial S)} \leq C \| V \|^2_{H^1(S)} \leq C \| v \|^2_{H^1(Q)} \leq C \{ \| y^{1/2} v(x, y) \|^2_{L^2(Q)} + \| y^{1/2} \partial_x v(x, y) \|^2_{L^2(Q)} \}.
\]

**Lemma 7.11** Let \( u \in H^{2,1}_\beta(Q) \) with \( 0 \leq \beta < 1/2 \). Let \( b_1 > a_1 > 0 \), \( h_2 > 0 \) and \( Q_1 = [a_1, b_1] \times [0, h_2] \). Then we have
\[
\| y^{1/2} D^\alpha u(x, y) - \bar{u}_h(x, y) \|^2_{L^2(Q_1)} \leq C h_2^{3-2|\alpha|} (h_2/a_1)^{2|\alpha|} \| u \|^2_{H^{2,1}_\beta(Q)}
\]
for \(|\alpha| \leq 1\).

**Proof.** For the derivative \( \partial_x (u(x, y) - \bar{u}_h(x, y)) \) we obtain
\[
\int_{a_1}^{b_1} \int_0^{h_2} \left[ y \partial_x u(x, y) - \partial_x u(x, h_2) \right]^2 \, dy \, dx
\]
\[\leq h_2 \int_0^{h_2} \left[ y \partial_x u(x, y) - \partial_x u(x, h_2) \right]^2 \, dy \, dx
\]
\[\leq \frac{h_2^2 h_1^{1-2\beta}}{2} \int_0^{b_1} \int_0^{h_2} y^2 \partial_y \partial_x u(x, y) \, dy \, dx
\]
\[\leq \frac{h_2^2 h_1^{1-2\beta}}{2} \int_0^{b_1} \int_0^{h_2} \left( y^2 \partial_x \partial_y u(x, y) \right) \, dy \, dx
\]
\[\leq \frac{h_2^2 h_1^{1-2\beta}}{2} \int_0^{b_1} \int_0^{h_2} \left( x y^2 \partial_x \partial_y u(x, y) \right) \, dy \, dx.
\]
CHAPTER 7. APPROXIMATION IN COUNTABLY NORMED SPACES

For the derivative $\partial_y (u(x, y) - \bar{u}_{h_2}(x, y))$ we obtain due to Lemma 7.8

$$
\int_{a_1}^{b_1} \int_0^{h_2} y \left( \partial_y u(x, y) - \frac{1}{h_2} \int_0^{h_2} \partial_z u(x, y) \, dz \right)^2 \, dy \, dx \\
\leq C h_2^{1-2\beta} \int_{a_1}^{b_1} \int_0^{h_2} \left( y^{1+\beta} \partial_y^2 u(x, y) \right)^2 \, dy \, dx.
$$

Finally, we obtain

$$
\int_{a_1}^{b_1} \int_0^{h_2} (y(u(x, y) - \bar{u}_{h_2}(x, y))^2 \, dy \, dx \\
= \int_{a_1}^{b_1} \int_0^{h_2} y \left( \int_0^{h_2} \partial_z u(x, z) - \bar{u}_{h_2}(x, z) \, dz \right)^2 \, dy \, dx \\
\leq \int_0^{h_2} y \int_y^{h_2} z^{-1} \, dz \, dy \int_{a_1}^{b_1} \int_0^{h_2} z (\partial_z u(x, z) - \bar{u}_{h_2}(x, z))^2 \, dz \, dx.
$$

We have

$$
\int_0^{h_2} y \int_y^{h_2} z^{-1} \, dz \, dy = \int_0^{h_2} y (\log h_2 - \log y) \, dy \\
= \frac{1}{2} h_2^2 \log h_2 - \frac{1}{2} h_2^2 \log h_2 - \frac{1}{4} h_2^2 = \frac{1}{4} h_2^2
$$

and therefore, we obtain

$$
\int_{a_1}^{b_1} \int_0^{h_2} (y(u(x, y) - \bar{u}_{h_2}(x, y))^2 \, dy \, dx \leq C h_2^{3-2\beta} \int_{a_1}^{b_1} \int_0^{h_2} \left( y^{1+\beta} \partial_y^2 u(x, y) \right)^2 \, dy \, dx.
$$

Let $I_y$ be the linear interpolation operator with respect to the variable $y$ on the interval $[0, h]$, e.g., $I_{y,u}(x, y) := \frac{h}{h_2} u(x, h) + \frac{h_2 - h}{h_2} u(x, 0) = u(x, h) - (1 - \frac{h}{h_2}) \int_0^h \partial_z u(x, z) \, dz$.

**Lemma 7.12** Let $0 \leq \beta < 1/2$, $h > 0$ and $u \in H_3^{1,1}(Q)$. Then there holds

$$
\| y^{1/2} D^\alpha \frac{x}{h} u(h, y) - I_y u(h, y) \|_{L_2([0,h]^2)} \leq C h^{1-2\beta} \| u \|_{H_3^{1,1}(Q)} \tag{7.63}
$$

for $0 \leq |\alpha| \leq 1$. 

\[ \square \]
Due to Lemma 7.8 we have
\[
\int_0^h \int_0^h y \left( \partial_y \frac{x}{h} (u(h, y) - I_y u(h, y)) \right)^2 \, dy \, dx = \int_0^h \int_0^h y \left( \partial_y u(h, y) - \frac{1}{h} \int_0^h \partial_z u(h, z) \, dz \right)^2 \, dy \, dx \leq C h^{1-\beta} \int_0^h \int_0^h \left( y^{1+\beta} \partial_y^2 u(h, y) \right)^2 \, dy \, dx \leq C h^{1-\beta} \int_0^h \int_0^h \left( \partial_y^2 u(x, y) \right)^2 \, dy \, dx \leq C h^{1-\beta} \int_0^h \int_0^h \left( \partial_y^2 u(x, y) \right)^2 \, dy \, dx
\]
Due to Lemma 7.9 we have
\[
\int_0^h \int_0^h y \left( \partial_y \frac{x}{h} (u(h, y) - I_y u(h, y)) \right)^2 \, dy \, dx \leq C h^{1-\beta} \int_0^h \int_0^h \left( \partial_y^2 u(x, y) \right)^2 \, dy \, dx + C h^{1-\beta} \int_0^h \int_0^h y^{2+\beta} \left( \int_x^h \partial_z \partial_y^2 u(z, y) \, dz \right)^2 \, dy \, dx \leq C h^{1-\beta} \int_0^h \int_0^h \left( \partial_y^2 u(x, y) \right)^2 \, dy \, dx + C h^{1-\beta} \int_0^h \int_0^h y^{2+\beta} \left( \int_x^h \partial_z \partial_y^2 u(z, y) \, dz \right)^2 \, dy \, dx
\]
there holds
\[
\int_0^h \int_0^h y \left( \partial_y \frac{x}{h} (u(h, y) - I_y u(h, y)) \right)^2 \, dy \, dx \leq C h^{1-\beta} \|u\|_{H^3; (0, h^2)}^2
\]
Analogously to the proof in Lemma 7.11 we have
\[
\|y^{1/2} \frac{x}{h} (u(h, y) - I_y u(h, y))\|_{L^2((0, h)^2)}^2 \leq \frac{1}{4} h^2 \|y^{1/2} \frac{x}{h} \partial_y (u(h, y) - I_y u(h, y))\|_{L^2((0, h)^2)}^2
\]
Finally, we obtain
\[
\int_0^h \int_0^h y \left( \partial_y \frac{x}{h} (u(h, y) - I_y u(h, y)) \right)^2 \, dy \, dx = \frac{1}{h} \int_0^h y (u(h, y) - I_y u(h, y))^2 \, dy \]
Due to Lemma 7.9 we can bound this term by
\[
\int_0^h \int_0^h y \left( \partial_y \frac{x}{h} (u(h, y) - I_y u(h, y)) \right)^2 \, dy \, dx \leq \frac{h^{2-\beta}}{1-\beta} \int_0^h \left( y^\beta \partial_y u(h, y) \right)^2 \, dy \leq \frac{h^{2-\beta}}{1-\beta} \left( \int_0^h \left( \int_x^h \partial_z \partial_y u(z, y) \, dz \right)^2 \, dy \right) \leq \frac{8 h^{1-\beta}}{1-2\beta} \left( \int_0^h \left( \int_x^h \partial_z \partial_y u(z, y) \, dz \right)^2 \, dy \right)
\]
For the first term we obtain the bound
\[
\int_{h/2}^{h} \int_{0}^{y} y^{2\beta} \left( \int_{x}^{h} \partial_{z} \partial_{y} u(z, x) \, dz \right)^{2} \, dy \, dx
\]
\[
\leq \int_{h/2}^{h} \int_{0}^{h} \int_{x}^{h} z^{-2} \, dz \, \int_{x}^{h} \left( z y^{\beta} \partial_{z} \partial_{y} u(z, x) \right)^{2} \, dz \, dy \, dx
\]
\[
\leq \int_{h/2}^{h} \left( x^{-1} - h^{-1} \right) \, dx \, \int_{0}^{h} \int_{0}^{h} \left( z y^{\beta} \partial_{z} \partial_{y} u(z, x) \right)^{2} \, dz \, dy
\]
\[
\leq \int_{0}^{h} \int_{0}^{h} \left( x y^{\beta} \partial_{x} \partial_{y} u(z, x) \right)^{2} \, dy \, dx.
\]

\[\square\]

**Lemma 7.13** Let $0 \leq \beta < 1/2$ and $u \in H^{1,1}_{\beta}(Q)$. Then, there holds
\[
\|u\|_{H^{1/2}([0,h])^2} \leq C \, h^{1/2-\beta} \|u\|_{H^{1,1}_{\beta}([0,h])^2} + C \, h^{-1/2} \|u\|_{L^2([0,h])^2}. \tag{7.64}
\]

If $\text{supp} \, u \subseteq [0, h]^2$, then there holds
\[
\|u\|_{H^{1/2}(Q)} \leq C \, h^{1/2-\beta} \|u\|_{H^{1,1}_{\beta}(Q)}. \tag{7.65}
\]

**Proof.** Let $\chi(\theta) \in C^1([0, \pi/2])$ be a cut-off function, i.e.,
\[
\chi(\theta) = \begin{cases} 
1, & 0 \leq \theta \leq \pi/6 \\
0, & \pi/3 \leq \theta \leq \pi/2
\end{cases}
\]
and $0 \leq \chi(\theta) \leq 1$. Let $R(\theta) \in C^0([0, \pi/2])$ a function such that
\[
[0, h]^2 = \{(r, \theta) : 0 \leq \theta \leq \pi/2, \quad 0 \leq r \leq R(\theta)\}.
\]

As a consequence we have $h \leq R(\theta) \leq \sqrt{2}h$.

\[
R(\theta) = \begin{cases} 
\frac{h}{\cos \theta}, & 0 \leq \theta \leq \pi/4 \\
\frac{h}{\cos(\pi/2 - \theta)}, & \pi/4 \leq \theta \leq \pi/2
\end{cases}
\]

Firstly, we can decompose
\[
\|u\|_{H^{1/2}([0,h])^1} \leq \|\chi(\theta)u\|_{H^{1/2}([0,h])^1} + \|\tilde{\chi}(\theta)u\|_{H^{1/2}([0,h])^1}
\]
with $\tilde{\chi}(\theta) := 1 - \chi(\theta)$. Due to Lemma 7.10 we have
\[
\|\chi(\theta)u\|_{H^{1/2}([0,h])^1}^2 \leq C \left\{ \|y^{1/2} \chi u\|_{L^2([0,h])^2}^2 + \|y^{1/2} \partial_x (\chi u)\|_{L^2([0,h])^2}^2 + \|y^{1/2} \partial_y (\chi u)\|_{L^2([0,h])^2}^2 \right\} \tag{7.66}
\]
and
\[
\|\tilde{\chi}(\theta)u\|_{H^{1/2}([0,h])^1}^2 \leq C \left\{ \|x^{1/2} \tilde{\chi} u\|_{L^2([0,h])^2}^2 + \|x^{1/2} \partial_x (\tilde{\chi} u)\|_{L^2([0,h])^2}^2 + \|x^{1/2} \partial_y (\tilde{\chi} u)\|_{L^2([0,h])^2}^2 \right\}. \tag{7.67}
\]

Due to the symmetry of the decomposition it is sufficient to show, that (7.66) is bounded by $\|u\|_{H^{1,1}_{\beta}([0,h])^2}^2$. 
We have $0 \leq \chi \leq 1$ and $y^{1/2} \leq h^{1/2}$ for $(x, y) \in [0, h]^2$. Therefore, we have
\[\|y^{1/2}x\|_{L^2([0,h]^2)}^2 \leq h \|u\|_{L^2([0,h]^2)}^2.\]

Furthermore, we have
\[
\|y^{1/2}\partial_x(\chi u)\|_{L^2([0,h]^2)} \leq \|y^{1/2}u(x, y)\partial_x(\chi)\|_{L^2([0,h]^2)} + \|y^{1/2}\chi \partial_x u\|_{L^2([0,h]^2)},
\]
\[
\|y^{1/2}\partial_y(\chi u)\|_{L^2([0,h]^2)} \leq \|y^{1/2}u(x, y)\partial_y(\chi)\|_{L^2([0,h]^2)} + \|y^{1/2}\chi \partial_y u\|_{L^2([0,h]^2)}.
\]

First, we can estimate
\[
\|y^{1/2}\chi \partial_\theta u\|_{L^2([0,h]^2)} \leq h^{1/2-\beta}\|y^{\beta}\partial_\theta u\|_{L^2([0,h]^2)}.
\]

Second, due to $\chi|_{[\pi/3, \pi/2]} = 0$, we have
\[
y^{1/2}\chi(\theta) = r^{1/2}\sin^{1/2}\chi(\theta) \leq \frac{\sin \pi/3}{\cos \pi/3} r^{1/2}\cos \theta \chi(\theta) = \sqrt{3}x^{1/2}\chi(\theta), \quad \forall \theta \in [0, \pi/2]
\]
and therefore
\[
\|y^{1/2}\chi \partial_x u\|_{L^2([0,h]^2)} \leq \sqrt{3}\|x^{1/2}\partial_x u\|_{L^2([0,h]^2)} \leq \sqrt{3}h^{1/2-\beta}\|x^{\beta}\partial_x u\|_{L^2([0,h]^2)}.
\]

Using $\partial_x = \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta$ and $\partial_y = \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta$ we obtain
\[
\|y^{1/2}u(x, y)\partial_x(\chi)(\theta)\|_{L^2([0,h]^2)} = \|y^{1/2}u(x, y)\frac{\sin \theta}{r} \partial_\theta(\chi)(\theta)\|_{L^2([0,h]^2)} \leq C\|r^{-1/2}u(x, y)\|_{L^2([0,h]^2)}
\]
and
\[
\|y^{1/2}u(x, y)\partial_y(\chi)(\theta)\|_{L^2([0,h]^2)} = \|y^{1/2}u(x, y)\frac{\cos \theta}{r} \partial_\theta(\chi)(\theta)\|_{L^2([0,h]^2)} \leq C\|r^{-1/2}u(x, y)\|_{L^2([0,h]^2)}.
\]

Then, we obtain
\[
\|r^{-1/2}u(x, y)\|_{L^2([0,h]^2)}^2 \leq \frac{\pi}{2} \int_0^{\pi/2} \int_0^{R(\theta)} r^{-1/2}u^2(r, \theta) r \, dr \, d\theta
\]
\[
= \frac{\pi}{2} \int_0^{\pi/2} \int_0^{R(\theta)} (u(r, \theta) - u(R(\theta), \theta))^2 \, dr \, d\theta
\]
\[
\leq 2 \int_0^{\pi/2} \int_0^{R(\theta)} \left( \int_0^{R(\theta)} \partial_z u(z, \theta) \, dz \right)^2 \, dr \, d\theta + 2 \int_0^{\pi/2} R(\theta) u^2(R(\theta), \theta) \, d\theta
\]
\[
\leq 2 \int_0^{\pi/2} \int_0^{R(\theta)} \int_0^{R(\theta)} z^{1-2\beta} \, dz \int_0^{R(\theta)} z \left( \frac{\beta}{2} \partial_z u(z, \theta) \right)^2 \, dz \, dr \, d\theta
\]
\[
+ 2 \int_0^{\pi/2} R(\theta) u^2(R(\theta), \theta) \, d\theta
\]
\[
\leq 2 \int_0^{\pi/2} \frac{R^{1-2\beta}(\theta)}{1-2\beta} \int_0^{R(\theta)} r \left( \frac{\beta}{2} \partial_r u(r, \theta) \right)^2 \, dr \, d\theta + 2 \int_0^{\pi/2} R(\theta) u^2(R(\theta), \theta) \, d\theta
\]
\[
\leq 2C \frac{h^{1-2\beta}}{1-2\beta} \|r^{\beta}\partial_r u(r, \theta)\|_{L^2([0,h]^2)}^2 + 2 \int_0^{\pi/2} R(\theta) u^2(R(\theta), \theta) \, d\theta.
\]
Here, we have used

\[
\int_0^{R(\theta)} \int_r^{R(\theta)} z^{-1-2\beta} \, dz \, dr = \int_0^{R(\theta)} \left( \frac{R^{-2\beta}(\theta) - r^{-2\beta}}{-2\beta} \right) \, dr = \frac{R^{1-2\beta}(\theta)}{-2\beta(1-2\beta)}.
\]

Due to \( \partial_r = \frac{x}{r} \partial_x + \frac{y}{r} \partial_y = \cos \theta \partial_x + \sin \theta \partial_y \) we obtain

\[
\|x^\beta \partial_x u(r, \theta)\|_{L^2([0,h]^2)} \leq \|x^\beta \cos \theta \partial_x u\|_{L^2([0,h]^2)} + \|x^\beta \sin \theta \partial_y u\|_{L^2([0,h]^2)}
\]

\[
\leq \|x^\beta \partial_x u\|_{L^2([0,h]^2)} + \|y^\beta \partial_y u\|_{L^2([0,h]^2)}
\]

\[
\leq C|u|_{H^{1,1}([0,h]^2)}.
\]

If \( \text{supp } u \subset [0,h]^2 \) then we have \( \int_0^{\pi/2} R(\theta) u^2(R(\theta), \theta) \, d\theta = 0 \) and (7.65) is proved. Otherwise we have

\[
\int_0^{\pi/2} R(\theta) u^2(R(\theta), \theta) \, d\theta = 2 \int_0^{\pi/2} \int_{R(\theta)/2}^{R(\theta)} u^2(R(\theta), \theta) \, dr \, d\theta
\]

\[
= 2 \int_0^{\pi/2} \int_{R(\theta)/2}^{R(\theta)} (u(R(\theta), \theta) - u(r, \theta) + u(r, \theta))^2 \, dr \, d\theta
\]

\[
\leq 2 \int_0^{\pi/2} \int_{R(\theta)/2}^{R(\theta)} \left( \int_r^{R(\theta)} \partial_z u(z, \theta) \, dz \right)^2 \, dr \, d\theta + 2 \int_0^{\pi/2} \int_{R(\theta)/2}^{R(\theta)} u^2(r, \theta) \, dr \, d\theta
\]

\[
\leq C|u|^2_{H^{1,1}([0,h]^2)} + 2 \int_0^{\pi/2} \frac{2}{R(\theta)} \int_{R(\theta)/2}^{R(\theta)} r u^2(r, \theta) \, dr \, d\theta
\]

\[
\leq C|u|^2_{H^{1,1}([0,h]^2)} + C \, h^{-1} \|u\|_{L^2([0,h]^2)}^2.
\]

Now, we have proved (7.64).

**Lemma 7.14** Let \( 0 \leq \beta < 1/2, u \in H^{2,1}_\beta(Q) \) and let

\[
\begin{align*}
  u^D(x, y) &:= \begin{cases} 
  u(x, y) - \frac{x}{h} u(h, y) - \frac{y}{h} u(x, h) + \frac{x+y}{h^2} u(h, h), & (x, y) \in [0,h]^2 \\
  0, & \text{elsewhere}
  \end{cases} \\
  \phi^D(x, y) &:= \begin{cases} 
  \frac{h-x}{h} \frac{h-y}{h} \phi_0, & \text{if } (x, y) \in [0,h]^2 \\
  0, & \text{elsewhere}
  \end{cases}
\end{align*}
\]

with \( \phi_0 = 0 \), if \( u(x, 0) = u(0, y) = 0, \forall x, y \in [0,1] \), or \( \phi_0 := \frac{1}{h^2} \int_0^h \int_0^h u(x, y) \, dy \, dx \) otherwise.

Then, there holds

\[
u^D - \phi^D \in H^{1,1}_\beta(Q) \quad \text{and} \quad \|u^D - \phi^D\|_{H^{1,1}_\beta(Q)} \leq C\|u\|_{H^{2,1}_\beta(Q)} \quad (7.68)
\]

with \( C > 0 \) independent of \( u \) and \( h \).
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Proof. First, we note that \( u^D - \phi^D |_{x=h} = 0, y \in [0, h] \) and \( u^D - \phi^D |_{y=h} = 0, x \in [0, h] \).

Therefore, we can write

\[
\int_0^h \int_0^h (u^D - \phi^D)^2 \, dy \, dx = \int_0^h \int_0^h (u^D(x, y) - \phi^D(x, y) - (u^D(h, y) - \phi^D(h, y)))^2 \, dy \, dx
\]

\[
= \int_0^h \int_0^h \left( \int_x^h \partial_x(u^D(z, y) - \phi^D(z, y)) \, dz \right)^2 \, dy \, dx
\]

\[
\leq \frac{h^{1-2\beta}}{1 - 2\beta} \int_0^h \int_0^h x^{2\beta} \left( \partial_x(u^D(x, y) - \phi^D(x, y)) \right)^2 \, dy \, dx
\]

Now, we have to investigate the first derivatives of \( u^D - \phi^D \). We can write

\[
\partial_x(u^D - \phi^D) = \partial_x u(x, y) - \frac{1}{h} u(h, y) - \frac{y}{h} \partial_x u(x, h) + \frac{1}{h} \frac{y}{h} u(h, h) + \frac{1 - y}{h} \phi_0
\]

\[
= \frac{h - y}{h} \partial_x u(x, y) - \frac{1}{h} \frac{h - y}{h} \partial_x u(x, h) + \frac{y}{h} \int_x^h \partial_z u(z, y) \, dz - \frac{y}{h} \int_x^h \partial_z \partial_z u(x, z) \, dz
\]

\[
+ \frac{1}{h^2} \frac{y}{h} \int_x^h \partial_z \partial_z u(z, w) \, dw \, dz + \frac{1}{h} \frac{y}{h} \int_x^h \partial_z u(x, z) \, dz
\]

\[
- \frac{1}{h} \frac{1 - y}{h} u(x, y) + \frac{1 - y}{h} \phi_0.
\]

Investigating the individual terms we obtain first

\[
\int_0^h \int_0^h x^{2\beta} \left( \frac{h - y}{h} \partial_x u(x, y) \right)^2 \, dy \, dx \leq \int_0^h \int_0^h (x^{\beta} \partial_x u(x, y))^2 \, dy \, dx
\]

and second

\[
\int_0^h \int_0^h x^{2\beta} \left( \frac{h - y}{h} \int_x^h \partial_z u(z, y) \, dz \right)^2 \, dy \, dx
\]

\[
\leq \frac{1}{h^2} \int_0^h \int_0^h x^{2\beta} \left( \int_x^h \partial_z u(z, y) \, dz \right)^2 \, dy \, dx
\]

\[
\leq \frac{1}{h^2} \int_0^h \int_0^h x^{2\beta} \int_x^h z^{-2\beta} \, dz \int_x^h \left( z^{\beta} \partial_z u(z, y) \right)^2 \, dz \, dx
\]

\[
\leq \frac{1}{h^2} \int_0^h x^{2\beta} \int_x^h z^{-2\beta} \, dz \int_0^h \left( z^{\beta} \partial_z u(z, y) \right)^2 \, dz \, dy
\]

\[
= \frac{1}{2(1 + 2\beta)} \int_0^h \int_0^h \left( x^{\beta} \partial_x u(x, y) \right)^2 \, dy \, dx
\]

and third

\[
\int_0^h \int_0^h x^{2\beta} \left( \frac{y}{h} \int_y^h \partial_z \partial_z u(x, z) \, dz \right)^2 \, dy \, dx
\]

\[
\leq \frac{1}{h^2} \int_0^h \int_0^h x^{2\beta} y^2 \int_y^h z^{-2} \, dz \int_y^h \left( z \partial_z \partial_z u(x, z) \right)^2 \, dz \, dy \, dx
\]

\[
\leq \frac{1}{h^2} \int_0^h y^2 (y^{-1} - h^{-1}) \int_y^h \left( x^{\beta} y \partial_z \partial_y u(x, y) \right)^2 \, dy \, dx
\]

\[
= \frac{1}{6} \int_0^h \int_0^h \left( x^{\beta} y \partial_z \partial_y u(x, y) \right)^2 \, dy \, dx
\]
and fourth
\[ \int_0^h \int_0^h x^{2\beta} \left( \frac{1}{h} \int_x^h \int_y^h \partial_z \partial_w u(z, w) \, dw \, dz \right)^2 \, dy \, dx \]
\[ = \frac{1}{h^4} \int_0^h \int_0^h x^{2\beta} y^2 \left( \int_x^h \int_y^h \partial_z \partial_w u(z, w) \, dw \, dz \right)^2 \, dy \, dx \]
\[ \leq \frac{1}{h^4} \int_0^h \int_0^h x^{2\beta} y^2 \int_x^h \int_y^h z^{-2\beta} w^{-2} \, dw \, dz \int_x^h \int_y^h \left( z^\beta \partial_z \partial_w u(z, w) \right)^2 \, dw \, dz \, dy \, dx \]
\[ \leq \frac{1}{h^4} \frac{h^2}{2(1 + 2\beta)} \frac{h^2}{6} \int_0^h \int_0^h \left( z^\beta \partial_z \partial_w u(z, w) \right)^2 \, dw \, dz \]

and fifth
\[ \int_0^h \int_0^h x^{2\beta} \left( \frac{1}{h} \int_x^h \int_y^h \partial_z u(x, z) \, dz \right)^2 \, dy \, dx \]
\[ \leq h^{-4 + 2\beta} \int_0^h \int_0^h y^2 \left( \int_y^h \partial_z u(x, z) \, dz \right)^2 \, dy \, dx \]
\[ \leq h^{-4 + 2\beta} \int_0^h \int_0^h y^2 \int_y^h y^{-2\beta} \int_y^h \left( z^\beta \partial_z u(x, z) \right)^2 \, dz \, dy \, dx \]
\[ = \frac{1}{3(4 - 2\beta)} \int_0^h \int_0^h \left( y^\beta \partial_y u(x, y) \right)^2 \, dy \, dx \]

and finally, if \( \phi_0 = \frac{1}{h^2} \int_0^h \int_0^h u(x, y) \, dy \, dx \)
\[ \int_0^h \int_0^h x^{2\beta} \left( \frac{1}{h} \int_x^h \int_y^h (u(x, y) - \phi_0) \right)^2 \, dy \, dx \]
\[ \leq h^{-2 + 2\beta} \int_0^h \int_0^h \left( \int_0^h \int_0^h (u(x, y) - u(z, y) + u(z, y) - u(z, w)) \, dw \, dz \right)^2 \, dy \, dx \]
\[ = h^{-6 + 2\beta} \int_0^h \int_0^h \left( \int_0^h \int_0^h \left\{ \int_x^z \partial_p u(p, y) \, dp + \int_y^z \partial_q u(z, q) \, dq \right\} \, dw \, dz \right)^2 \, dy \, dx \]
\[ \leq \frac{2}{1 - 2\beta} \left( \int_0^h \int_0^h \left( x^\beta \partial_x u(x, y) \right)^2 \, dy \, dx + \int_0^h \int_0^h \left( y^\beta \partial_y u(x, y) \right)^2 \, dy \, dx \right), \]

otherwise, if \( \phi_0 = 0 \) and \( u \mid_{[0,1] \times \{0\}} = u \mid_{\{0\} \times [0,1]} = 0 \), we have
\[ \int_0^h \int_0^h x^{2\beta} \left( \frac{1}{h} \int_x^h \int_y^h (u(x, y) - \phi_0) \right)^2 \, dy \, dx \]
\[ \leq h^{-2} \int_0^h \int_0^h x^{2\beta} (u(x, y) - u(0, y))^2 \, dy \, dx \]
\[ \leq h^{-2} \int_0^h \int_0^h x^{2\beta} \int_0^x z^{-2\beta} \int_0^z (z^\beta \partial_z u(z, y))^2 \, dz \, dy \, dx \]
\[ \leq \frac{1}{2(1 - 2\beta)} \int_0^h \int_0^h \left( x^\beta \partial_x u(x, y) \right)^2 \, dy \, dx. \]
Combining all terms we obtain
\[ \| \partial_x (u^D - \phi^D) \|^2_{L^2(Q)} \leq C \| u \|^2_{H_\beta^{1,1}(Q)}. \]

Due to the symmetry of \( u^D(x, y) - \phi^D(x, y) \) in \( x \) and \( y \), we also have
\[ \| \partial_y (u^D - \phi^D) \|^2_{L^2(Q)} \leq C \| u \|^2_{H_\beta^{1,1}(Q)}. \]

\[ \Box \]

**Lemma 7.15** Let \( 0 \leq \beta < 1/2 \) and let \( 0 < h \leq 1 \). Let \( u \in H^{1/2}(Q) \cap H_\beta^{1,1}(Q) \) with \( u \mid_{[0,1] \times \{0\} \equiv 0, u \mid_{\{0\} \times [0,1]} \equiv 0 \). Then there holds
\[ \| u \|^2_{H^{1/2}(Q)} \leq \| u \|^2_{H_\beta^{1,1}(Q)} + \frac{1}{h^2} \| u \|^2_{H^{1}(\{0,1\})} \]
\[ + \frac{h^{1-2\beta}}{(1 - 2\beta)^2} \left( \| u \|^2_{H_\beta^{1,1}(\{0,1\})} + \| y^\beta \partial_y u \|^2_{L^2([1-\beta,1-\beta] \times [0,1])} + \| x^\beta \partial_x u \|^2_{L^2([0,1] \times [1-\beta,1-\beta])} \right). \] (7.69)

**Proof.** Due to Definition 7.2 there holds
\[ \| u \|^2_{H^{1/2}(Q)} = \| u \|^2_{H_\beta^{1,1}(Q)} + \int_0^1 \int_0^1 \frac{|u(x, y)|^2}{\partial(x, y)} dy dx \]
\[ = \| u \|^2_{H_\beta^{1,1}(Q)} + \int_0^1 \int_0^x y^{-1} u^2(x, y) dy dx + \int_0^1 \int_0^y x^{-1} u^2(x, y) dy dx. \]
Then, we can split the integrals
\[ \int_0^1 \int_0^x y^{-1} u^2(x, y) dy dx + \int_0^1 \int_0^y x^{-1} u^2(x, y) dy dx = \int_0^1 \int_0^h y^{-1} u^2(x, y) dy dx + \int_0^1 \int_0^h x^{-1} u^2(x, y) dy dx \]
\[ + \int_0^1 \int_h^x y^{-1} u^2(x, y) dy dx + \int_0^1 \int_h^y x^{-1} u^2(x, y) dy dx \]
\[ + \int_0^1 \int_h^x y^{-1} u^2(x, y) dy dx + \int_h^1 \int_h^x x^{-1} u^2(x, y) dy dx. \]
First, we have
\[ \int_h^1 \int_h^x y^{-1} u^2(x, y) dy dx + \int_h^1 \int_h^y x^{-1} u^2(x, y) dy dx \leq h^{-1} \int_h^1 \int_h^1 u^2(x, y) dy dx. \]
Second, we have
\[ \int_0^1 \int_0^x y^{-1} u^2(x, y) dy dx = \int_0^h \int_0^x y^{-1} \left( \int_0^y \partial_x u(x, z) dz \right)^2 dy dx \]
\[ \leq \int_0^h \int_0^x y^{-1-2\beta} \int_0^y (z^\beta \partial_x u(x, z))^2 dz dy dx \]
\[ \leq \int_0^h \int_0^x y^{-2\beta} dy \int_0^x (z^\beta \partial_x u(x, z))^2 dz dx \]
\[ = \int_0^h \frac{x^{1-2\beta}}{(1 - 2\beta)^2} \int_0^x (z^\beta \partial_x u(x, z))^2 dz dx \]
\[ \leq \frac{h^{1-2\beta}}{(1 - 2\beta)^2} \int_0^h \int_0^x (z^\beta \partial_x u(x, z))^2 dz dx. \]
Analogously, we have
\[
\int_0^1 \int_0^h y^{-1}u^2(x, y) \, dy \, dx = \int_0^1 \int_0^h y^{-1}\left(\int_0^y \partial_z u(x, z) \, dz\right)^2 \, dy \, dx \\
\leq \frac{h^{1-2\beta}}{(1-2\beta)^2} \int_0^h \int_0^h (z^{2\beta}\partial_z u(x, z))^2 \, dz \, dx.
\]

Lemma 7.16 Let \( u \in H^{1,1}_\beta(Q) \) with \( 0 \leq \beta < 1/2 \), let \( h > 0 \) and let
\[
\bar{u}_h(x, y) := u(x, h) - \left(1 - \frac{y}{h}\right) \int_0^h \partial_z u(x, z) \, dz = I_y u(x, y).
\]
Then, we have
\[
\|y^{2\beta}\partial_y (u(x, y) - \bar{u}_h(x, y))\|_{L^2([h, 1] \times [0, h])} \leq C \|u\|_{H^{1,1}_\beta(Q)}.
\]
(7.70)

Proof. We have
\[
\int_0^1 \int_0^h y^{2\beta}\left(\partial_y u(x, y) - \frac{1}{h} \int_0^h \partial_z u(x, z) \, dz\right)^2 \, dy \, dx \\
\leq 2\int_0^1 \int_0^h y^{2\beta}(\partial_y u(x, y))^2 \, dy \, dx + \frac{2}{h^2} \int_0^1 \int_0^h y^{2\beta}\left(\int_0^h \partial_z u(x, z) \, dz\right)^2 \, dy \, dx \\
\leq 2\|u\|_{H^{1,1}_\beta(Q)}^2 + \frac{2}{h^2} \int_0^h y^{2\beta} \, dy \int_0^h z^{-2\beta} \, dz \int_0^1 \int_0^h z^{2\beta}(\partial_z u(x, z))^2 \, dz \, dx \\
= 2\|u\|_{H^{1,1}_\beta(Q)}^2 + \frac{2}{h^2} \frac{h^{1+2\beta} h^{1-2\beta}}{1 - 2\beta} \int_0^1 \int_0^h z^{2\beta}(\partial_z u(x, z))^2 \, dz \, dx \\
= \left(2 + \frac{2}{1 - 4\beta^2}\right) \|u\|_{H^{1,1}_\beta(Q)}^2.
\]

Lemma 7.17 Let \( 0 \leq \beta < 1/2, h > 0 \) and \( u \in H^{1,1}_\beta(Q) \). Then there holds
\[
\|\frac{x}{h} (u(h, y) - I_y u(h, y))\|_{H^{1,1}_\beta([0, h]^2)}^2 \leq C \|u\|_{H^{1,1}_\beta(Q)}^2.
\]
(7.71)
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Proof. We have analogously to Lemma 7.16

\[ \|y^\beta \partial_y^x I_\beta(u(h, y) - I_y u(h, y))\|_{L^2([0, h]^2)}^2 \]
\[ = \frac{h}{3} \int_0^h y^{2\beta} \left( \partial_y u(h, y) - \frac{1}{h} \int_0^h \partial_z u(h, z) \right)^2 dy \]
\[ \leq C \frac{h}{3} \int_0^h (y^\beta \partial_y u(h, y))^2 dy \]
\[ = C \frac{2}{3} \int_0^h y^{2\beta} \int_{h/2}^h (\partial_y u(h, y) - \partial_y u(x, y) + \partial_y u(x, y))^2 dx dy \]
\[ \leq C \int_0^h y^{2\beta} \int_{h/2}^h \left( \int_x^h \partial_z \partial_y (z, y) \right. \left. dz \right)^2 dx dy + C \int_0^h y^{2\beta} \int_{h/2}^h (\partial_y u(x, y))^2 dx dy \]
\[ \leq C \int_0^h \int_{h/2}^h \int_x^h z^{-2} dz \int_x^h (y^{3\beta} \partial_x \partial_y (z, y))^2 dz dx dy + C \int_0^h \int_0^h (y^\beta \partial_y u(x, y))^2 dx dy \]
\[ \leq C \int_0^h \int_{h/2}^h \int_x^h z^{-2} dz \int_x^h (xy^{\beta} \partial_x \partial_y (z, y))^2 dz dx dy + C \int_0^h \int_0^h (y^\beta \partial_y u(x, y))^2 dx dy \]
\[ = C (\log 2 - \frac{1}{2}) \int_0^h \int_0^h (xy^{\beta} \partial_x \partial_y (z, y))^2 dy dx + C \int_0^h \int_0^h (y^\beta \partial_y u(x, y))^2 dx dy \]
\[ \leq C |u|_{H^{2,1}_\beta(Q)}^2 \]

and we have

\[ \|x^\beta \partial_x^x I_\beta(u(h, y) - I_y u(h, y))\|_{L^2([0, h]^2)}^2 \]
\[ = \frac{h^{-1+2\beta}}{1+2\beta} \int_0^h (u(h, y) - I_y u(h, y))^2 dy \]
\[ = \frac{h^{-1+2\beta}}{1+2\beta} \int_0^h \left( \int_0^y \partial_z (u(h, z) - I_z u(h, z)) dz \right)^2 dy \]
\[ \leq \frac{h^{-1+2\beta}}{1+2\beta} \int_0^h \int_0^y z^{-2\beta} dz \int_0^y z^{2\beta} \left( \partial_z (u(h, z) - \frac{1}{h} \int_0^h \partial_u u(h, w) dw \right)^2 dz dy \]
\[ \leq \frac{h^{-1+2\beta}}{1+2\beta} \int_0^h \int_0^y z^{-2\beta} dz \int_0^y y^{2\beta} \left( \partial_y u(h, y) - \frac{1}{h} \int_0^h \partial_z u(h, z) dz \right)^2 dy \]
\[ = \frac{h^{-1+2\beta}}{1+2\beta} \int_0^h \int_0^y y^{2\beta} \left( \partial_y u(h, y) - \frac{1}{h} \int_0^h \partial_z u(h, z) dz \right)^2 dy \]
\[ = \frac{h}{(1-4\beta^2)(2-2\beta)} \int_0^h y^{2\beta} \left( \partial_y u(h, y) - \frac{1}{h} \int_0^h \partial_z u(h, z) dz \right)^2 dy \]
\[ \leq C |u|_{H^{2,1}_\beta(Q)}^2. \]

\[ \Box \]

7.4 Proof of Theorem 7.4

Due to Lemma 7.1 we have \( u \in C^0(\bar{Q}\setminus\{(0, 0)\}) \), therefore, the point evaluation of \( u(x, y) \) is possible.
Let $0 \leq \beta < 1/2$. Let $I_y$ be the linear interpolation operator with respect to the variable $y$ on the interval $[0, h_1]$, e.g., $I_y u(x, y) := \frac{y}{h_1} u(x, h_1) + \frac{h_1 - y}{h_1} u(x, 0) = u(x, h_1) - (1 - \frac{y}{h_1}) \int_0^{h_1} \partial_y u(x, z) \, dz$.

We split $u$ into $u = u^A + u^B + u^C + u^D$ according to

\[
\begin{align*}
    u^A(x, y) &= \begin{cases} 
        u(x, y), & \text{if } (x, y) \in [h_1, 1]^2, \\
        I_y u(x, y), & \text{if } (x, y) \in [h_1, 1] \times [0, h_1], \\
        I_x u(x, y), & \text{if } (x, y) \in [0, h_1] \times [h_1, 1], \\
        \frac{x}{h_1} I_y u(h_1, y) + \frac{h_1 - x}{h_1} u(0, h_1), & \text{if } (x, y) \in [0, h_1]^2,
    \end{cases} \\
    u^B(x, y) &= \begin{cases} 
        u(x, y) - I_y u(x, y), & \text{if } (x, y) \in [h_1, 1] \times [0, h_1], \\
        \frac{x}{h_1} u(h_1, y) - \frac{x}{h_1} I_y u(h_1, y), & \text{if } (x, y) \in [0, h_1]^2,
    \end{cases} \\
    u^C(x, y) &= \begin{cases} 
        u(x, y) - I_x u(x, y), & \text{if } (x, y) \in [0, h_1] \times [h_1, 1], \\
        \frac{y}{h_1} u(h_1, y) - \frac{y}{h_1} I_y u(h_1, y), & \text{if } (x, y) \in [0, h_1]^2,
    \end{cases} \\
    u^D(x, y) &= \begin{cases} 
        u(x, y) + \frac{x}{h_1} \frac{y}{h_1} u(h_1, h_1) - \frac{x}{h_1} u(h_1, 0) - \frac{h_1 - y}{h_1} u(0, h_1), & \text{if } (x, y) \in [0, h_1]^2, \\
        0, & \text{otherwise}.
    \end{cases}
\end{align*}
\]

We will construct the spline $\phi \in S^{n,1}(Q^*_n)$ approximating $u$ by constructing splines $\phi^A, \phi^B, \phi^C, \phi^D \in S^{n,1}(Q^*_n)$ which are approximating $u^A, u^B, u^C, u^D$ such that $\phi = \phi^A + \phi^B + \phi^C + \phi^D$.

**Construction of $\phi^A$**

First, we note that $u$ and $u^A$ are identical on $[h_1, 1]^2$, therefore also $\phi$ and $\phi^A$ are identical on $[h_1, 1]^2$. In the following, we usually will not distinguish between $u$ and $u^A$ (or $\phi$ and $\phi^A$, respectively) if we discuss the approximation on $[h_1, 1]^2$.

Due to the definition of $u^A$ we define

\[
\phi^A(x, y) = \frac{x}{h_1} u(h_1, h_1) + \frac{y}{h_1} u(h_1, 0) + \frac{h_1 - y}{h_1} u(0, h_1)
\]

and we have $(u^A - \phi^A)|_{[0,h_1]^2} \equiv 0$.

Due to Lemma 7.3, we get polynomials $\psi_{kl} \in P_{p_k+1}(R_{kl})$ on the edge-strip $\{(x, y) | y \leq h_1 \leq x \leq 1, 0 \leq y \leq h_1\}$ (and also on $\{(x, y) | 0 \leq x \leq h_1, h_1 \leq y \leq 1\}$, respectively) which coincide in the corner points with $u$ and which fulfill with suitable $s_k$

\[
\|D^\alpha (u^A - \psi_{kl})\|_{L^2(R_{kl})}^2 \leq C \frac{x_{k-1}^{2(1-\beta)} + x_{l-1}^{2(1-\alpha_2)}}{x_{k-1}^{2(1-\beta) - 2\alpha_1}} \left( \frac{\lambda_1}{2} \right)^{2(s_k+1-\alpha_1)} |u|_{H^s_{\beta}+3,1}^2 (Q), \tag{7.72}
\]

Due to Theorem 7.6 (cf. [46, Theorem 4.1]), we get polynomials $\psi_{kl} \in P_{p_k+1}(R_{kl})$ with $1 \leq s_k \leq p_k + 1$ and $0 \leq \alpha_1, \alpha_2 \leq 1$ for the inner elements $R_{kl}$ ($2 \leq k, l \leq n$) ($D^\alpha = \partial_{x}^{\alpha_1} \partial_{y}^{\alpha_2}$):
CHAPTER 7. APPROXIMATION IN COUNTABLY NORMED SPACES

For the construction of a continuous spline \( \phi(x, y) \in S^{p-1}(Q^*_n) \) we have to investigate the jumps of \( \psi_{kl} \) from \( R_{kl} \) to its neighbor elements. Let \( 1 \leq k \leq n - 1, 1 \leq l \leq n \). Then \( \psi_{kl} \) and \( \psi_{k+1,l} \) coincide with \( u(x, y) \) in the points \((x_k, x_l)\) and \((x_k, x_{l-1})\). Therefore, the difference \( \tilde{w}_{kl}^r = (\psi_{k+1,l} - \psi_{kl})|_{\gamma_{kl}^r} \) vanishes in the endpoints of the common side \( \gamma_{kl}^r = \{x_k\} \times [x_{l-1}, x_l] \) of \( R_{kl} \) and \( R_{k+1,l} \). \( \tilde{w}_{kl}^r \) is a polynomial of degree \( \leq p_l \) in \( y \). Due to Lemma 7.6 there is a polynomial \( w_{kl}^r(x, y) \) of degree \( p_l \) in \( y \) and degree 1 in \( x \) such that for \( x_{l-1} \leq y \leq x_l \) we have \( w_{kl}^r(x, y) = \tilde{w}_{kl}^r(y) \). The polynomial \( w_{kl}^r \) vanishes on the other sides of \( R_{kl} \). For \( 0 \leq |\alpha| \leq 1 \) we have

\[
\|D^\alpha w_{kl}^r(x, y)\|_{L^2(R_{kl})}^2 \leq C h_k^{1-2\alpha_1} h_l^{2(1-\alpha_2)} \|\partial_y \tilde{w}_{kl}^r(y)\|_{L^2(\gamma_{kl}^r)}^2
\]

\[
\leq C h_k^{1-2\alpha_1} h_l^{2(1-\alpha_2)} \left( \|\partial_y (\psi_{kl} - u)\|_{L^2(\gamma_{kl}^r)}^2 + \|\partial_y (\psi_{k+1,l} - u)\|_{L^2(\gamma_{kl}^r)}^2 \right). \tag{7.74}
\]

Analogously, we have \( \tilde{w}_{kl}^o = (\psi_{k,l+1} - \psi_{kl})|_{\gamma_{kl}^o} \) on the common side \( \gamma_{kl}^o = [x_{k-1}, x_k] \times \{x_l\} \) of \( R_{kl} \) and \( R_{k,l+1} \). \( \psi_{k,l+1} \) and \( \psi_{k+1,1} \) (\( 1 \leq k \leq n - 1 \)) coincide with \( u(x, y) \) in the endpoints of the common side. Due to construction, \( \psi_{k,1} \) and \( \psi_{k+1,1} \) are linear in \( y \), i.e., there is no jump between \( \psi_{k,1} \) and \( \psi_{k+1,1} \). The same result holds for \( \psi_{1,1} \) and \( \psi_{1,l+1} \) (\( 1 \leq l \leq n - 1 \)). Therefore, we have \( w_{kl}^o = 0 \) for \( 1 \leq k \leq n - 1 \) and \( w_{kl}^o = 0 \) for \( 1 \leq l \leq n - 1 \).

![Figure 7.2: Element to element interfaces](image)

\[
\phi_{kl} = \psi_{kl} + w_{kl}^r + w_{kl}^o. \tag{7.75}
\]

Due to construction, on the common side \( \gamma_{kl}^r = \{x_k\} \times [x_{l-1}, x_l] \) of \( R_{k+1,l} \) and \( R_{kl} \) for \( 1 \leq k \leq n - 1, 1 \leq l \leq n \) there holds

\[
(\phi_{k+1,l} - \phi_{kl})|_{\gamma_{kl}^r} = (\psi_{k+1,l} - \psi_{kl} + w_{k+1,l}^r - w_{kl}^r + w_{k+1,l}^o - w_{kl}^o)|_{\gamma_{kl}^r} = 0. \tag{7.76}
\]

Analogously, we have at the common side \( \gamma_{kl}^o = [x_{k-1}, x_k] \times \{x_l\} \) of \( R_{k,l+1} \) and \( R_{kl} \) for \( 1 \leq k \leq n, 1 \leq l \leq n - 1 \)

\[
(\phi_{k,l+1} - \phi_{kl})|_{\gamma_{kl}^o} = (\psi_{k,l+1} - \psi_{kl} + w_{k,l+1}^r - w_{kl}^r + w_{k,l+1}^o - w_{kl}^o)|_{\gamma_{kl}^o} = 0. \tag{7.77}
\]
Therefore, there is a continuous function $\phi$ with $\phi|_{R_{kl}} = \phi_{kl}$, i.e., $\phi \in S^{p,1}(Q_{kl})$ and

$$
\|D^o(u^A - \phi^A)\|_{L^2(Q)}^2 = \sum_{k,l=1}^{n} \|D^o(u^A - \psi_{kl} - w_{k+1}^0 - w_{kl}^0)\|_{L^2(R_{kl})}^2
\leq
3 \sum_{k,l=1}^{n} \|D^o(u^A - \psi_{kl})\|_{L^2(R_{kl})}^2
+ \sum_{k=1}^{n-1} \sum_{l=2}^{n} \|D^o w_{k+1}^0\|_{L^2(R_{kl})}^2
+ \sum_{k=2}^{n} \sum_{l=1}^{n-1} \|D^o w_{kl}^0\|_{L^2(R_{kl})}^2
\leq
3 \sum_{k,l=1}^{n} \|D^o(u^A - \psi_{kl})\|_{L^2(R_{kl})}^2
+ C \sum_{k=1}^{n-1} \sum_{l=2}^{n} h^0_{l-1,2} h^0_{l,2} \left( \|\partial_\gamma (\psi_{kl} - u^A)\|_{L^2(\gamma_{kl})}^2 + \|\partial_y (\psi_{k+1,l} - u^A)\|_{L^2(\gamma_{kl})}^2 \right)
+ C \sum_{k=2}^{n} \sum_{l=1}^{n-1} h^0_{k,1} h^0_{l,2} \left( \|\partial_\gamma (\psi_{kl} - u^A)\|_{L^2(\gamma_{kl})}^2 + \|\partial_y (\psi_{k,l+1} - u^A)\|_{L^2(\gamma_{kl})}^2 \right)
(7.78)

In the following, we estimate the terms on the side $\gamma_{kl}^0$. The terms on $\gamma_{kl}$ can be estimated analogously. The terms $\|\partial_\gamma (\psi_{kl} - u^A)\|_{L^2(\gamma_{kl})}^2$ for the strips at the edges have to be investigated separately.

Due to Lemma 7.7, we have for $2 \leq k, l \leq n$

$$
h^0_{k,1} h^0_{l,2} \|\partial_\gamma (\psi_{kl} - u^A)\|_{L^2(\gamma_{kl})}^2 \leq
C h^0_{k,1} h^0_{l,2} \left( \|\partial_\gamma (\psi_{kl} - u^A)\|_{L^2(R_{kl})}^2 + h^0_{l-1} \|\partial_y (\psi_{kl} - u^A)\|_{L^2(R_{kl})}^2 \right)
\leq
C \left( h^0_{k,1} h^0_{l,2} \|\partial_\gamma (\psi_{kl} - u^A)\|_{L^2(R_{kl})}^2 + h^0_{k,1} h^0_{l,2} \|\partial_y (\psi_{kl} - u^A)\|_{L^2(R_{kl})}^2 \right).
$$

Similarly, we get with $h_k = \sigma h_{k+1} = \lambda \sigma x_k$ (cf. (7.12))

$$
h^0_{k,1} h^0_{l,2} \|\partial_\gamma (\psi_{kl} - u^A)\|_{L^2(\gamma_{kl})}^2 \leq
C \left( h^0_{k,1} h^0_{l,2} \|\partial_\gamma (\psi_{kl} - u^A)\|_{L^2(R_{kl})}^2 + h^0_{k,1} h^0_{l,2} \|\partial_y (\psi_{kl} - u^A)\|_{L^2(R_{kl})}^2 \right).
$$

Therefore, using Theorem 7.6 and $h_k = \lambda x_{k-1}$, see (7.12), we get

$$
h^0_{k,1} h^0_{l,2} \|\partial_\gamma (\psi_{kl} - u^A)\|_{L^2(\gamma_{kl})}^2 \leq
C \left( h^0_{k,1} h^0_{l,2} \|\partial_\gamma (\psi_{kl} - u^A)\|_{L^2(R_{kl})}^2 + h^0_{k,1} h^0_{l,2} \|\partial_y (\psi_{kl} - u^A)\|_{L^2(R_{kl})}^2 \right).
$$

(7.79)
Hence

\[
\sum_{k=2}^{n} \sum_{l=2}^{n} h_k^{2(1-\alpha)} h_l^{1-2\alpha} \|\partial_x (\psi_{kl} - u^A)\|^2_{L^2(\gamma^k_0)} \leq \nabla
\]

\[
\leq C \sum_{k=2}^{n} \sum_{l=2}^{n} x_{k-1}^{2(1-\alpha-\beta)} x_l^{-2\alpha} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+1}_\beta(Q)}
\]

\[
+ C \sum_{k=2}^{n} \sum_{l=2}^{n} x_{k-1}^{-2\alpha+2} x_l^{1-2\alpha+\beta} \frac{\Gamma(p_l - s_l + 1)}{\Gamma(p_l + s_l + 1)} \left(\frac{\lambda}{2}\right)^{2s_l} |u|^2_{H^{s_l+1}_\beta(Q)}. \quad (7.80)
\]

The terms in (7.80) are of the same structure as (7.73). Due to the symmetry of (7.80) in \(\alpha_1, \alpha_2\) (or \(k, l\) resp.), it is sufficient to investigate the behavior of the term

\[
\sum_{k=2}^{n} \sum_{l=2}^{n} x_{k-1}^{2(1-\alpha-\beta)} x_l^{-2\alpha} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+1}_\beta(Q)}
\]

for \((\alpha_1, \alpha_2) = (0, 0), (1, 0)\) and \((0, 1)\).

For \(\alpha_1 = 0\) and \(\alpha_2 = 0\) we obtain with \(x_k = \sigma^{-k}\) (see (7.11))

\[
= (n-1)\sigma^{2(1-\beta)(n-1)} \sum_{k=2}^{n} \sigma^{2(1-\beta)(-k+2)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+1}_\beta(Q)}.
\]

For \(\alpha_1 = 1\) and \(\alpha_2 = 0\) we obtain

\[
= (n-1)\sigma^{-(n-1)-2\beta} \sum_{k=2}^{n} \sigma^{-(k-2)\beta} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+1}_\beta(Q)}.
\]

For \(\alpha_1 = 0\) and \(\alpha_2 = 1\) we obtain

\[
= (n-1)\sigma^{-2(n-1)} \sum_{k=2}^{n} \sigma^{-2(1-\beta)(n-1)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+1}_\beta(Q)}.
\]

\[
= (n-1)\sigma^{-2\beta(n-1)} \sum_{k=2}^{n} \sigma^{-2(1-\beta)(-k+2)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+1}_\beta(Q)}.
\]

Here, we have used

\[
\sum_{l=2}^{n} x_l^{-2} = \sum_{l=2}^{n} \sigma^{-2(n-l+1)} = \frac{\sigma^{-2(n-1)} - 1}{1 - \sigma^2} \leq C \sigma^{-2(n-1)}. \quad (7.81)
\]
Now, we are investigating the terms $\|\partial_x (\psi_k - u^A)\|^2_{L^2(\gamma^1_1)}$ for the strips on the edges in (7.78) separately. Using (7.38) in Lemma 7.3, $h_1 = x_1$ and $|u|^2_{H^{s+2.1}_\beta(Q)} \leq |u|^2_{H^{s+3.1}_\beta(Q)}$, we have

$$
\sum_{k=2}^n h_{1}^{4(1-\alpha)} h_{1}^{-2\alpha} \|\partial_x (\psi_k - u^A)\|^2_{L^2(\gamma^1_1)} \\
\leq C \sum_{k=2}^n h_{1}^{4(1-\alpha)} h_{1}^{-2\alpha} h_{1}^{-1} x_{k-1}^{-2} \left( h_{1}^{2(1-\beta)} + x_{k-1}^{2(1-\beta)} \right) \Gamma(p_k - s_k + 1) \Gamma(p_k + s_k + 1) \left( \frac{\lambda}{2} \right)^{2s_k} |u|^2_{H^{s+2.1}_\beta(Q)} \\
\leq C \sum_{k=2}^n x_{k-1}^{-\alpha} x_{1}^{2(1-\beta-\alpha)} \Gamma(p_k - s_k + 1) \Gamma(p_k + s_k + 1) \left( \frac{\lambda}{2} \right)^{2s_k} |u|^2_{H^{s+1}_\beta(Q)} \\
+ C \sum_{k=2}^n x_{k-1}^{-\alpha} x_{1}^{-\alpha} \Gamma(p_k - s_k + 1) \Gamma(p_k + s_k + 1) \left( \frac{\lambda}{2} \right)^{2s_k} |u|^2_{H^{s+3.1}_\beta(Q)}. \quad (7.82)
$$

The terms in (7.82) are of the form of (7.72) and they are the leading terms in the summation of (7.80). Therefore, the bounds obtained for (7.80) are also applicable to (7.82) and (7.72).

Due to $u \in B^s_{\beta}(Q)$, we have $u \in H^{s+3.1}_\beta(Q)$, and due to (7.23), we have $|u|^2_{H^{s+3.1}_\beta(Q)} \leq Cd^{s+2}\Gamma(s_k + 3)$. Therefore, we obtain with $\varrho = \max(1, \lambda)$

$$
\sum_{k=2}^n \sigma^{2(1-\beta)(-k+2)} \Gamma(p_k - s_k + 1) \Gamma(p_k + s_k + 1) \left( \frac{\lambda}{2} \right)^{2s_k} |u|^2_{H^{s+1}_\beta(Q)} \\
\leq \sum_{k=2}^n \sigma^{2(1-\beta)(-k+2)} \Gamma(p_k - s_k + 1) \Gamma(p_k + s_k + 1) \left( \frac{\varrho d}{2} \right)^{2(s_k + 2)} \Gamma(s_k + 3)^2. \quad (7.83)
$$

On the other hand, setting

$$
F(\alpha, d) := \frac{(1 - \alpha)^{1-\alpha}}{(1 + \alpha)^{1+\alpha}} \left( \frac{\alpha d}{2} \right)^{2\alpha} \quad (7.84)
$$

there holds for $s_k = \alpha p_k$ (see [46, 51, 67])

$$
\frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left( \frac{\varrho d}{2} \right)^{2(s_k + 2)} \left( \Gamma(s_k + 3) \right)^2 \leq F(\alpha, \varrho d)^{p_k \alpha} P_k^{2(1-2 \alpha) + \varrho d p_k} = CF(\alpha, \varrho d)^{p_k},
$$

where $p_k = \max(2, \lceil \mu(k-1) \rceil + 1)$ for $\mu > 0$ ($k = 2, \ldots, n$). Setting $\alpha_k = \max(1/p_k, \alpha_{\min})$ ($k = 2, \ldots, n$) with $\alpha_{\min} = \frac{2}{\sqrt{4+\varrho^2 d^2}}$, we get from (7.83) the estimate

$$
\sum_{k=2}^n \sigma^{2(1-\beta)(-k+2)} \Gamma(p_k - s_k + 1) \Gamma(p_k + s_k + 1) \left( \frac{\lambda}{2} \right)^{2s_k} |u|^2_{H^{s+3.1}_\beta(Q)} \leq C \sum_{k=2}^n \sigma^{2(1-\beta)(-k+2)} F(\alpha_k, \varrho d)^{p_k} p_k^5.
$$

Let $F_{\min} := F(\alpha_{\min}, \varrho d)$ and

$$
\mu > \frac{2(1-\beta) \log \sigma}{\log F_{\min}} \quad (7.86)
$$

and let $k_0$ be defined by the equation $p_{k_0} = \left[ \frac{1}{\alpha_{\min}} \right] + 1$. Then, $k_0$ is bounded yielding

$$
p_{k_0} = \lceil \mu(k_0 - 1) \rceil \leq \frac{1}{\alpha_{\min}} + 2. \quad (7.87)
$$
Therefore, we can bound (7.85) by
\[
\sum_{k=2}^{n} \sigma^{2(1-\beta)(-k+2)} F(\alpha_k, q\delta) p_k^5 p_k^5 \leq \sum_{k=2}^{k_0} \sigma^{2(1-\beta)(-k+2)} F(1/p_k, q\delta) p_k^5 p_k^5 + \sum_{k=k_0+1}^{n} \tilde{\sigma}^{2(1-\beta)(-k+2)} (F_{\min})^5 p_k^5. \tag{7.88}
\]
There holds
\[
\sigma^{2(1-\beta)(1-k)} F p_k \leq \sigma^{2(1-\beta)} F_{\min}^{(\mu(k-1))} \leq C \sigma^{2(1-\beta)} \left( \frac{F_{\min}^{\mu}}{\sigma^{2(1-\beta)}} \right) k. \tag{7.89}
\]
and \( F_{\min} < \sigma^{2(1-\beta)} \), due to \( F_{\min} < 1 \) (see (7.86) and Theorem 5.1 in [46]). Therefore, we have \( q := \frac{F_{\min}^{\mu}}{\sigma^{2(1-\beta)}} < 1 \) and \( \sum_{k=k_0}^{n} q^k k^5 < \infty \). Hence, the series on the right hand side in (7.88)
\[
\sum_{k=2}^{k_0} \sigma^{2(1-\beta)(-k+2)} F(1/p_k, q\delta) p_k^5 p_k^5 + \sum_{k=k_0+1}^{n} \sigma^{2(1-\beta)(-k+2)} (F_{\min})^5 p_k^5 \tag{7.90}
\]
is bounded.
Altogether, this yields the estimates
\[
\|u^A - \phi^A\|_{L^2(Q)} \leq C \sqrt{n-1} \sigma^{(1-\beta)(n-1)}, \tag{7.91}
\]
\[
\|u^A - \phi^A\|_{H^1(Q)} \leq C \sqrt{n-1} \sigma^{-\beta(n-1)}. \tag{7.92}
\]
Interpolation [9] yields
\[
\|u^A - \phi^A\|^2_{H^{1/2}(Q)} \leq \|u^A - \phi^A\|_{L^2(Q)} \cdot \|u^A - \phi^A\|_{H^1(Q)} \leq C (n-1) \sigma^{(1-\beta)(n-1)} \sigma^{-\beta(n-1)} = C (n-1) \sigma^{(1-2\beta)(n-1)}. \tag{7.93}
\]
Therefore, we have
\[
\|u^A - \phi^A\|^2_{H^{1/2}(Q)} \leq C e^{-2bn}
\]
for \( n \geq n_0 \) with a fixed integer \( n_0 \) and with \( b = -((\log n_0)/n_0 + (1 - 2\beta) \log \sigma) > 0 \). For the number of degrees of freedom \( N = \dim S^{n,1}(Q^n) \) we have
\[
N = \left( 2 + \sum_{i=1}^{n-1} (1 + \max(2, [\mu_i] + 1)) \right)^2 \leq \left( 2(n+1) + \frac{n(n-1)}{2} \right)^2 \leq C n^4.
\]
Finally, we get
\[
\|u^A - \phi^A\|_{H^{1/2}(Q)} \leq C e^{-b \frac{1}{2} N}. \tag{7.94}
\]

**Construction of \( \phi_B, \phi_C \)**

Due to Lemma 7.11 and Lemma 7.12, we have that
\[
\sum_{|\alpha| \leq 1} \| y^{1/2} D^\alpha u_B(x,y) \|^2_{L^2(Q)} = \sum_{|\alpha| \leq 1} \| y^{1/2} D^\alpha u_B(x,y) \|^2_{L^2([0,1] \times [0, h_1])} \leq C h_1^{1-2\beta} \| u \|^2_{H^{3,1}_B(Q)}. \tag{7.95}
\]
Therefore, we can choose \( \phi_B \equiv 0 \) and obtain by the trace theorem
\[
\| u_B - \phi_B \|^2_{H^{1/2}(Q)} \leq C h_1^{1-2\beta} \| u \|^2_{H^{3,1}_B(Q)}. \tag{7.96}
\]
The result for \( u_C \) follows by symmetry.
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Construction of $\phi^D$

Let $\phi^D(x,y)$ be

$$\phi^D(x,y) = \begin{cases} \frac{h-1-x}{h_1} y \phi_0, & \text{if } (x,y) \in [0,h_1]^2, \\ 0, & \text{elsewhere} \end{cases}$$

with $\phi_0 = 0$ if $u|_{[0,1] \times [0]} \equiv 0$, $u|_{[0] \times [0,1]} \equiv 0$, or with $\phi_0 := \frac{1}{h^2} \int_0^h \int_0^h u(x,y) \, dy \, dx$ otherwise. Due to Lemma 7.14, we have $u^D - \phi^D \in H_0^{1,1}(Q)$ and $\|u^D - \phi^D\|_{H_0^{1,1}(Q)} \leq C\|u\|_{H_0^{2,1}(Q)}$ with $C$ independent of $u$ and $h_1$.

Due to Lemma 7.13, there holds for $u^D - \phi^D$ that

$$\|u^D - \phi^D\|_{H_0^{1,2}(Q)} \leq C h_1^{1/2-\beta} \|u_D - \phi_D\|_{H_0^{1,1}(Q)}.$$  

Therefore, combining all approximation results, we obtain

$$\|u - \phi\|_{H_0^{1,2}(Q)} \leq \|u^A - \phi^A\|_{H_0^{1,2}(Q)} + \|u^B\|_{H_0^{1,2}(Q)} + \|u^C\|_{H_0^{1,2}(Q)} + \|u^D - \phi^D\|_{H_0^{1,2}(Q)} \leq C e^{-b \sqrt[\alpha]{\eta}}. \quad (7.97)$$

\[\square\]

7.5 Proof of Theorem 7.5

Let $u = u^A + u^B + u^C + u^A$ be the decomposition used in the proof of Theorem 7.4 and $\phi = \phi^A + \phi^B + \phi^C + \phi^D$ the spline approximating $u$. Let $h_1 = \sigma^{n-1}$. Due to Lemma 7.15 we have the following bound

$$\|u - \phi\|_{H_0^{1,2}(Q)}^2 \leq \|u - \phi\|_{H_0^{1,2}(Q)}^2 + h_1^{-1} \|u - \phi\|_{L^2([0,h_1]^2)}^2 + \frac{h_1^{1-2\beta}}{(1-2\beta)^2} \|u - \phi\|_{H_0^{1,1}([0,h_1]^2)}^2 \leq \frac{h_1^{1-2\beta}}{(1-2\beta)^2} \left( \|y^2 \partial_y (u - \phi)\|_{L^2([0,h_1]^2 \times [0,h_1])}^2 + \|x^2 \partial_x (u - \phi)\|_{L^2([0,h_1]^2 \times [0,h_1])}^2 \right).$$

Due to Theorem 7.4, we already have

$$\|u - \phi\|_{H_0^{1,2}(Q)}^2 \leq C (n-1) \sigma^{(1-2\beta)(n-1)}.$$

Due to the construction of $u - \phi$, we have $(u - \phi)|_{[h_1,1]^2} = (u^A - \phi^A)|_{[h_1,1]^2}$, i.e., due to (7.91), we have

$$h_1^{-1} \|u - \phi\|_{[h_1,1]^2}^2 = h_1^{-1} \|u^A - \phi^A\|_{[h_1,1]^2}^2 \leq C h_1^{-1} \sigma^{2(1-\beta)(n-1)} = C (n-1) \sigma^{(1-2\beta)(n-1)}.$$  

Due to (7.92), we have

$$\|y^2 \partial_y (u^A - \phi^A)\|_{L^2([0,h_1]^2 \times [0,h_1])}^2 \leq h_1^{2\beta} \|\partial_y (u^A - \phi^A)\|_{L^2([0,h_1]^2 \times [0,h_1])}^2 \leq h_1^{2\beta} \|u^A - \phi^A\|_{H^1(Q)}^2 \leq C h_1^{2\beta} (n-1) \sigma^{-2\beta (n-1)} = C (n-1),$$

and analogously $\|x^2 \partial_x (u^A - \phi^A)\|_{L^2([0,h_1]^2 \times [h_1,1])}^2 \leq C (n-1)$. 


Due to Lemma 7.16, we have
\[ \| y^2 \partial_y (u^B - \phi^B) \|_{L^2([h_1,1] \times [0,h_1])}^2 \leq C \| u \|_{H^{1,1}_\beta(Q)}^2 \]
and due to Lemma 7.17, we have
\[ |u^B - \phi^B|_{H^{1,1}([0,h_1])} \leq C \| u \|_{H^{1,1}_\beta(Q)}. \]
For \( u^C - \phi^C \) the analogous results are valid.
Due to Lemma 7.14, we have
\[ \| u^D - \phi^D \|_{H^{1,1}_\beta(Q)} \leq C \| u \|_{H^{1,1}_\beta(Q)}. \]
Fitting together, we obtain the assertion.

7.6 Numerical results

In this section we present numerical results for the boundary element Galerkin method of the Neumann problem of the Laplacian on the geometry given in Figure 7.1 with boundary data \( g(x) \equiv 1 \). For the pure \( h \) and \( p \) versions there hold the following error estimates for the Galerkin solution \( v_N \)
\[ \| v - v_N \|_{H^{1/2}(\Gamma)} \leq c_4 \left\{ \begin{array}{ll} h^{1/2-\varepsilon} & \text{for } h \text{ version} \\ p^{-1+2\varepsilon} & \text{for } p \text{ version} \end{array} \right\} \sim c_4 \left\{ \begin{array}{ll} N^{-1/4+\varepsilon} & \text{for } h \text{ version} \\ N^{-1/2+2\varepsilon} & \text{for } p \text{ version} \end{array} \right\} \]
(see Theorem 2.2 in [36] for the \( h \) version and [86] for the \( p \) version). We can extrapolate a numerical value of \( \| v \|_{H^{1/2}(\Gamma)} \approx 0.5160 \) for the energy norm.
Due to Theorem 7.2, we expect for the \( hp \) version with geometric mesh numerically an exponentially fast convergence. In Figure 7.3 we compare the errors for the different versions of the boundary element Galerkin method for the hypersingular integral equation (7.5) on \( \Gamma \) using the energy norm: The error curves are linear for the pure \( h \) and \( p \) versions and curved downward for the \( hp \) version with geometric mesh. The convergence of the latter is indeed exponential as Figure 7.4 indicates where we obtain nearly straight lines by plotting the logarithmic errors in energy norm against the fourth root of the number of unknowns. The computation of the Galerkin matrix has been done by transforming the hypersingular integrals into weakly singular integrals by integration by parts and computing the occurring four dimensional integrals analytically. For computations of Helmholtz problems, see [58].
Figure 7.3: BEM Galerkin errors for hypersingular integral equation on the triangle in $\mathbb{R}^3$

Figure 7.4: BEM Galerkin errors for hypersingular integral equation on the triangle in $\mathbb{R}^3$
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Table 7.1: Convergence rates (energy norm) for the hypersingular integral equation on the triangle
Bibliography


Notations

\[ \langle \cdot, \cdot \rangle \quad \text{Duality between } H^{1/2}(\Gamma) \text{ and } H^{-1/2}(\Gamma) \text{ or the specified abbreviation for } \langle \cdot, \cdot \rangle_X. \]

\[ \langle \cdot, \cdot \rangle_X \quad \text{Duality between } X \text{ and } X^* \text{ if } X \text{ is a Banach space or the scalar product in a Hilbert space } X. \]

\[ \| \cdot \|_X \quad \text{Norm in a Banach space } X. \]

\[ X^* \quad \text{Dual space if } X \text{ is a Banach space.} \]

\[ L(X,Y) \quad \text{Vector space of all bounded linear operators mapping } X \text{ into } Y, \]
\[ X, Y \text{ being Banach spaces. } L(X,Y) \text{ is endowed with the operator norm } \| \cdot \| := \| \cdot \|_{L(X,Y)}. \]

\[ \text{dist} \quad \text{dist}(y, X) := \text{dist}_Y(y, X) := \inf_{x \in X} \| x-y \|_Y \text{ where } y \in Y, X \subset Y; \]
\[ X, Y \text{ being Banach spaces.}\]

\[ (\cdot, \cdot)_n \quad \text{Scalar product in } H^n(\Omega) \text{ for } n \in \mathbb{N}_0. \]

\[ \| \cdot \|_n \quad \text{Norm in } H^n(\Omega) \text{ for } n \in \mathbb{N}_0. \]

\[ \Omega, \Omega_c \quad \text{Lipschitz domains, } \Omega_c = \mathbb{R}^n \setminus \bar{\Omega}. \]

\[ \Gamma_s, \Gamma_t \quad \text{Lipschitz boundaries, Signorini and transmission interface.} \]

\[ \Gamma_D, \Gamma_N, \Gamma_S \quad \text{Lipschitz boundaries with Dirichlet, Neumann and Signorini conditions imposed.} \]

\[ H^s(\Omega), H^s(\Gamma), \bar{H}^s(\Gamma) \quad \text{Sobolev spaces, see Definition 3.1 on page 16.} \]

\[ H^s_{loc}(\Omega_c) \quad \text{see Definition 5.1 on page 85} \]

\[ H(div; \Omega), H_{loc}(div; \Omega_c) \quad \text{see Definition 5.1 on page 85} \]

\[ V, K, K', W \quad \text{Integral operators, see Definition 3.2 on page 17 for the Laplacian and see Definition 3.8 on page 25 for the Lame operator.} \]

\[ S \quad \text{Steklov-Poincaré operator for the exterior domain, see Definition 4.8 on page 53,} \]
\[ S = \frac{1}{2}(W + (K' - I)V^{-1}(K - I)) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma); \]
\[ \text{for the interior domain, see Definition 3.5 on page 18,} \]
\[ S = \frac{1}{2}(W + (K' + I)V^{-1}(K + I)) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma). \]

\[ R \quad \text{inverse Steklov-Poincaré operator for the exterior domain, see Definition 5.4 on page 90,} \]
\[ R = \frac{1}{2}(V + (I + K)W^{-1}(I + K')) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma). \]

\[ H_h, L_h, H_h^{1/2}, H_h^{-1/2}, H_{s,h}^{1/2} \quad \text{Discrete subspaces of } H(div; \Omega), L^2(\Omega), H^{1/2}(\Gamma)/\mathbb{R}, H^{-1/2}(\Gamma), \]
\[ H^{1/2}(\Gamma)_s, \text{ see page 99.} \]

\[ T_h \quad \text{Regular triangulation of the domain } \Omega. \]

\[ \mathcal{E}_h \quad \text{Set of all edges in } T_h. \]

\[ E_h \quad \text{Equilibrium interpolation operator, see page 105.} \]
(L), (L_h) Problems (L), (L_h), see Definition 3.3 on page 18 and Definition 3.7 on page 20.
(S) Problem (S), see Definition 3.4 on page 18.
(P), (P_h) Problem (P), (P_h), see Definition 4.1 on page 54 and Definition 4.3 on page 56.
(P_I) Problem (P_I), see Definition 4.2 on page 55.
(P) Problem (P), see Definition 5.5 on page 91.
(M), (M_h) Problems (M), (M_h), see Definition 5.6 on page 92 and Definition 5.9 on page 100.
(M_I), (M_{h,I}) Problems (M_I), (M_{h,I}), see Definition 5.8 on page 98 and Definition 5.10 on page 104.