Approximation Methods for Convolution Operators
on the Real Line

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Abstract

This work is concerned with the applicability of several approximation methods (finite section method, Galerkin and collocation methods with maximum defect (discontinuous) splines for uniform or non uniform meshes) to operators belonging to the closed subalgebra of $\mathcal{L}(L^2(\mathbb{R}))$ generated by operators of multiplication by piecewise continuous functions in $\mathbb{R}$ and convolution operators also with piecewise continuous generating functions. The traditional techniques are not powerful enough to give an answer to this problem. So the strategy followed is to use Banach algebra techniques. To this end, a larger algebra of sequences is introduced, which contains the special sequences we are interested in. There is a direct relationship between the applicability of the approximation method for a given operator and the invertibility of the corresponding sequence in this algebra. Exploring this relationship, the methods of essentialization, localization and identification of the local algebras through construction of locally equivalent representations are used in order to derive useful invertibility criteria. Finally, some examples are presented, including explicit conditions for the applicability of finite section and Galerkin methods to Wiener-Hopf operators with piecewise continuous symbols and some other related operators (singular integral and Hankel operators).

Key words

Finite section method, Galerkin method, Collocation method, Convolution operators, Multiplication operators, Wiener-Hopf operators, Hankel operators.
To Ana, Sara
and Catarina
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Chapter 1

Introduction

Boundary integral equation methods are an important tool in the arsenal of the engineer or the applied mathematician when treating problems related to wave diffraction or thermoelastodynamics, for example. Of the resulting operators, the convolution type operators have an important role. The basis for a part of the theory of these operators was already developed in the twenties and thirties with the classical papers of F. Noether [25] (for the singular integral operator), N. Wiener and E. Hopf [13] (for the Wiener-Hopf operators) and S.G. Mikhlin [21] (the introduction of the concept of symbol for an operator). The study of approximation methods for operator equations started a little later, and in the forties and fifties the theory regarding operators of the form identity plus compact or identity plus operator of small norm appeared (see for example [18]). The main problem regarding approximation methods is that the invertibility of the associated operator is in general not sufficient to guarantee convergence of the approximation methods normally used (see for instance [28]). The question is how to identify the extra necessary conditions and obtain a set of sufficient conditions. Here it must be also mentioned the pioneering book of I. Gohberg and I. Feldman [15], where effective conditions for the applicability of the finite section method to equations involving Wiener-Hopf operators with continuous symbol were derived. It happens that the finite section method turned out to have an extraordinary importance for the theoretical point of view of numerical analysis (see for instance [28] Chapters 4 and 5). Also in that book, the authors made use of the deep analogy between Wiener-Hopf and Toeplitz operators, pointed out by M. Rosenblum [35] and M.G. Krein [20] a few years before (The last paper is a famous one, where it was also introduced factorization theory in the study of convolution operators). Meanwhile the theory of convolution operators and some algebras generated by convolution and/or multiplication operators were studied by R. Duduchava, R.G. Douglas, I. Gohberg and N. Krupnik (see for instance the books [9], [10] and [16]) in part by the use of local methods. In the field of numerical analysis, with a direct relationship to the present work, Galerkin and collocation approximation methods with discontinuous or continuous splines were studied by J. Elschner for Wiener-Hopf and Mellin operators with continuous symbol [13]. One of the main tools used was factorization theory.
CHAPTER 1. INTRODUCTION

The next important idea was introduced by A. Kozak in [19] where he “algebraized” the stability (convergence) problem. That is, he proved the equivalence between the stability problem and an invertibility problem in a suitably chosen Banach algebra. This equivalence indicated that algebraic techniques were useful in the study of approximation methods. He was able to show this in a very special situation: finite section method for Toeplitz operators (one dimensional and multidimensional) acting on $l^2$ spaces with continuous generating functions. The main idea was that such operators are operators of local type with respect to a “nice” system of projections. This fact permitted him to apply the local principle of Simonenko [41]. The problem was, that even for finite sections of Toeplitz operators with piecewise continuous generating functions such ideas are not applicable, because the operators are not of local type in that sense. Moreover, there is a variety of approximation processes (collocation, quadrature rules, harmonic approximation, etc) which do not fit in that notion of operator of local type mentioned above. So the question arose: “Is there a general way to handle such problems?” Nowadays it is well known that there is indeed a general procedure. It is aimed on showing that even for algebras which possess a trivial center it is often possible to obtain “effective” invertibility criteria. These procedure had a long development, which was started by B. Silbermann [38] and reached its conceptualization with the notion of lifting ideals and the Lifting Theorem presented in [17]. In an abstract way, these procedure consists of the following four steps:

(a) Algebraization: Find a unital Banach algebra $\mathcal{E}$ and a two-sided ideal $\mathcal{G} \subset \mathcal{E}$ such that the original stability problem becomes equivalent to an invertibility problem in the quotient algebra $\mathcal{E}/\mathcal{G}$.

(b) Essentialization: Find a unital subalgebra $\mathcal{F} \subset \mathcal{E}$ and an ideal $\mathcal{J} \subset \mathcal{F}$, both containing $\mathcal{G}$, such that $\mathcal{J}$ can be lifted, and such that if $\mathcal{A} \subset \mathcal{F}$ is the unital subalgebra containing the elements that interest us, the algebra $\mathcal{A}\mathcal{J} = (\mathcal{A}+\mathcal{J})/\mathcal{J}$ has a large center.

The most interesting property of the essentialization or lifting step is that there is no invertibility information lost when the algebra is factored. This has to do with the special properties of the ideal $\mathcal{J}$ that are necessary for the application of the Lifting Theorem (see [17, Theorem 1.8]).

(c) Localization: Use a localization principle to translate the invertibility problem in the algebra $\mathcal{A}\mathcal{J}$ to simpler local algebras.

(d) Identification: Find necessary and sufficient conditions for the invertibility of the elements in the local algebras. This can be done using for example the Two Projections Theorem (see for instance [17, Theorem 1.10]) or by the definition of isomorphisms between the local algebras and known algebras (of operators, for example).

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1 or, as is mentioned in this work, the problem of applicability of the approximation method.
In [17] this procedure was used for a wide set of approximation methods with regard to singular integral operators and some types of Mellin operators. S. Roch in [30] treated the infinite Galerkin method (i.e., the Galerkin method in which the spline space has infinite dimension) with smoothest splines and uniform meshes, for Wiener-Hopf operators with piecewise continuous symbols. It is in this line that the present work is situated. The aim is to find necessary and sufficient conditions for the applicability of the finite section method and (finite or infinite) Galerkin methods for operators in algebras generated by multiplication and convolution operators, with the possible addition of a flip.

There were two main difficulties with the application of the above procedure to the case under consideration. The first was to find the ideal $\mathcal{J}$ necessary in the essentialization step. Indeed until now, when dealing with operators defined on the real line, the sequence of projections related to the approximation method was never considered as a simple generator of the algebra, together with the constant sequences that represent the operators we are interested in applying the approximation to. This more abstract view led to a much enlarged algebra than usual, and in order that after essentialization, resulted an algebra with suitable commutation properties between its elements, it was necessary to extend the concept until now held, of possible ideals $\mathcal{J}$. After tackling successfully over the problem of the ideal $\mathcal{J}$, and after the localization step, the local algebras were not of the simplest kind, having 3 or even 4 generators. As a general theory for algebras generated by 3 or more generators does not exist (see [2]), it was necessary, case by case, or find an isomorphism between the local (sequence) algebras and known (usually operator) algebras, or find enough relationships between the different generators in order to again apply localization procedures and/or a flip elimination scheme to finally arrive at algebras generated by one element or two projections. For all local algebras where this last procedure was used, it was always necessary to calculate some (non trivial) local spectrum of a combination of generators.

In this work are derived conditions for the applicability of the finite section method and Galerkin method with uniform meshes and maximum defect splines to the elements of the algebra generated by multiplication and convolution operators with piecewise continuous symbol. The addition of a flip to the algebra is considered, permitting to include also Hankel type operators. It is also seen that the collocation method for the singular integral operator (at least) is included in the results. For non-uniform meshes, it was possible to obtain some results regarding piecewise polynomial splines and Wiener-Hopf operators with piecewise continuous symbol. These results are a generalization of some of J. Elschner’s results presented in [13], mainly those regarding the Galerkin method.

The work is organized as follows. In the next chapter are introduced the basic concepts from functional analysis and the technical results that will be needed further on. That chapter can be skipped, and be used as a reference while reading the others, at the reader’s discretion. The third chapter is dedicated to the finite section method, and the results are written explicitly for algebras containing the flip. The fourth chapter treats Galerkin and collocation methods with uniform meshes, and here are the most
important results from this work. The fifth chapter approaches the more complex subject of non-uniform meshes.

I would like to use this last lines of the introduction to thank all the people at the Fakultät für Mathematik, Technische Universität Chemnitz for their help and support, and make a special acknowledgment to Professor Dr. Bernd Silbermann, that besides the excellent working relationship, always managed to show his friendship. I would also like to refer my colleague Tilo Finck who, receiving me as a friend, helped solve many of the small problems always associated with studying in a foreign country. Finally, to my wife Ana, without whose love, comprehension and personal sacrifice I would not be able to arrive where I arrived, goes my deepest gratitude.

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Chapter 2

Some auxiliary material

2.1 About operators and sequences of operators in Hilbert spaces

Let $H$ be a Hilbert space with inner (and duality) product $(.,.)$ and norm $\|u\| = (u,u)^{\frac{1}{2}}$. The set of all bounded operators acting on $H$, $\mathcal{L} = \mathcal{L}(H)$, with the sum $(A+B)u = Au + Bu$ and the product $(AB)u = A(Bu)$, $u \in H$, together with the derivate norm $\|A\| = \sup_{\|u\|=1} \|Au\|$ and the involution given using the inner product forms a $C^*$-algebra, because $\|AA^*\| = \|A\|^2$ (see [36, Chapter 12]). Given an unbounded ordered index set $I$ we say that a sequence $(A_\tau)$ in $\mathcal{L}$ is uniformly bounded if $\sup_\tau \|A_\tau\| < +\infty$.

1 Define the following types of convergence for sequences of operators in $\mathcal{L}$.

**Definition 2.1.1.** The sequence $(A_\tau)$ tends (or converges) in the norm to $A \in \mathcal{L}$ if and only if $\|A_\tau - A\|$ converges to zero.

**Definition 2.1.2.** The sequence $(A_\tau)$ tends (or converges) strongly to $A \in \mathcal{L}$ if and only if $\|(A_\tau - A)u\|$ converges to zero for any $u \in H$.

**Definition 2.1.3.** The sequence $(A_\tau)$ tends (or converges) weakly to $A \in \mathcal{L}$ if and only if $(v,(A_\tau - A)u)$ converges to zero for any $u,v \in H$.

We will use the symbol “$\Rightarrow$” to indicate convergence in the norm, the symbol “$\rightarrow$” to indicate strong convergence for operator sequences or the usual convergence for scalar sequences, and the symbol “$\rightharpoonup$” to indicate weak convergence. It is not difficult to verify that norm convergence implies strong convergence, and that strong convergence implies weak convergence. The next results are also true.

**Lemma 2.1.1.** If $(A_\tau u)$ is a convergent sequence to zero in $H$ for each $u$ belonging to a dense subset of $H$ and $(A_\tau)$ is uniformly bounded, then $(A_\tau)$ converges strongly to zero.

1When the index set is clear from the context, we will avoid writing it explicitly.
Proof: Given \( u \in H \) and \( \epsilon > 0 \) choose \( n_\epsilon \), such that for \( n \geq n_\epsilon \) we have \( \| u - u_n \| \sup_{\tau} \| A_\tau \| < \frac{\epsilon}{2} \), with \( u_n \) in the dense subset of \( H \) and tending to \( u \). Then it is not difficult to see that there is a \( \tau_\epsilon \), such that
\[
\| A_\tau u \| \leq \| A_\tau u_n \| + \sup_{\tau} \| A_\tau \| \| u - u_n \| < \epsilon
\]
for \( \tau > \tau_\epsilon \), and this implies the strong convergence.

Lemma 2.1.2. If \((v, A_\tau u)\) converges to zero for each \( u \) and \( v \) belonging to a dense subset of \( H \) and \((A_\tau)\) is uniformly bounded, then \((A_\tau)\) converges weakly to zero.

Proof: For \( v \in H \) and \( u \) in the dense subset of \( H \), given \( \epsilon > 0 \), choose \( n_\epsilon \) such that for \( n \geq n_\epsilon \) we have \( \| v - v_n \| \sup_{\tau} \| A_\tau \| \| u \| < \frac{\epsilon}{2} \), with \( v_n \) in the dense subset of \( H \). Then it is possible to find a \( \tau_\epsilon \) such that for \( \tau \geq \tau_\epsilon \) we have
\[
| (v, A_\tau u) | \leq | (v_n, A_\tau u) | + | (v - v_n, A_\tau u) | < \epsilon.
\]
This means that \((v, A_\tau u)\) converges to zero for each \( u \) belonging to a dense subset of \( H \) and \( v \in H \). It is possible to apply now a similar reasoning considering \( u \) and \( v \) in \( H \) and thus conclude that the convergence is in fact for any \( u, v \in H \), which implies the weak convergence of \((A_\tau)\) to zero.

Theorem 2.1.3 (Banach-Steinhaus). If \((A_\tau u)\) is a convergent sequence in \( H \) for each \( u \in H \), then \( \sup_{\tau} \| A_\tau \| < +\infty \), the operator \( A \) defined by \( Au = \lim_{\tau \to +\infty} A_\tau u \) belongs to \( \mathcal{L} \) and \( \| A \| \leq \lim \inf_{\tau \to +\infty} \| A_\tau \| \).

Because we are interested in working with algebraic structures, it is important to answer the question “Is the limit of the product of sequences equal to the product of the limits?” and other related questions. The answer, as one can see in the next results, depends on the type of convergence. In the next results, \( A \) and \( B \) designate operators from \( \mathcal{L} \).

Lemma 2.1.4. If \( A_\tau \Rightarrow A \) then \( A_\tau^* \Rightarrow A^* \) and \((A_\tau)\) is uniformly bounded. If also \( B_\tau \Rightarrow B \) then \( A_\tau B_\tau \Rightarrow AB \).

Proof: The first assertion comes from the fact that for any operator \( T \) in \( \mathcal{L} \) we have \( \| T \| = \| T^* \| \). The uniform boundness of \((A_\tau)\) can be seen by writing \( \| A_\tau \| \leq \| A_\tau - A \| + \| A \| \). Finally it is possible to write
\[
\| AB - A_\tau B_\tau \| \leq \| A - A_\tau \| \| B \| + \| A_\tau \| \| B - B_\tau \|
\]
and this proves the last assertion.

Lemma 2.1.5. Let \((A_\tau)\) be uniformly bounded. If \( B_\tau \to 0 \) then \( A_\tau B_\tau \to 0 \). If \( A_\tau \to A \) and \( B_\tau \to B \) then \( A_\tau B_\tau \to AB \).
Proof: Consider $u \in H$ and write $\|A_r B_r u\| \leq \|A_r\| \|B_r u\|$ to prove the first part and $\|A B u - A_r B_r u\| \leq \|(A - A_r) B u\| + \|A_r\| \|(B - B_r) u\|$ to prove the second.

For the weak convergence, the limit of the product of two sequences is in general not equal to the product of the limits. But the following results about multiplication of strongly and weakly convergent sequences still hold.

Lemma 2.1.6. Let $(A_r)$ be an uniformly bounded sequence such that $A_r \rightharpoonup A$. Then $A_r B \rightharpoonup AB$ and $B A_r \rightharpoonup BA$. If moreover $A_r^* \rightarrow A^*$ and $B_r \rightarrow B$ then we have that $(A_r B_r)$ tends also weakly to $AB$.

Proof: The first results come almost directly from the definition of weak convergence. For the second we have for any $u, v \in H$

$$(v, A_r B_r u) = (A_r^* v, B_r u) - (A_r^* v, B u) + (A_r^* v, B u) = (A_r^* v, (B_r - B) u) + ((A_r^* - A^*) v, B u) + (A^* v, B u).$$

As

$$|((A_r^* v, (B_r - B) u)| \leq (A_r^* v, A_r^* v)^{\frac{1}{2}} ((B_r - B) u, (B_r - B) u)^{\frac{1}{2}} \rightarrow 0,$$

$$|((A_r^* - A^*) v, B u)| \leq ((A_r^* - A^*) v, (A_r^* - A^*) v)^{\frac{1}{2}} (B u, B u)^{\frac{1}{2}} \rightarrow 0$$

and $(A^* v, B u) = (v, AB u)$ the result follows.

Lemma 2.1.7. Let $(A_r)$ and $(B_r)$ be uniformly bounded sequences such that $A_r \rightharpoonup A$ and $B_r^* \rightarrow B^*$. Then $B_r A_r \rightharpoonup BA$.

Proof: We have, for any $u, v \in H$

$$(v, B_r A_r u) = (B_r^* v, A_r u) = (B^* v, A_r u) + ((B^* - B_r^*) v, A_r u),$$

with $(B^* v, A_r u) \rightarrow (B^* v, Au) = (v, BA u)$ and

$$((B^* - B_r^*) v, A_r u) \leq ((B^* - B_r^*) v, (B^* - B_r^*) v)^{\frac{1}{2}} (A_r u, A_r u)^{\frac{1}{2}}$$

tending to zero.

Let $K$ be the closed subset of $L$ formed by the compact operators. The following additional results follow (see e.g. [28]).

Lemma 2.1.8. If $K \in K$ and $A_r \rightharpoonup A$ then $KA_r \rightharpoonup KA$.

Lemma 2.1.9. If $A_r \rightarrow A$, $B_r^* \rightarrow B^*$ and $K \in K$ then $A_r K B_r \rightrightarrows AKB$. 

2.2 Multiplication and convolution operators

As usual, we let $L^2(\mathbb{R})$ and $l^2$ represent respectively the Hilbert spaces of Lebesgue integrable complex valued functions $u$ defined in $\mathbb{R}$ such that the norm\footnote{As it is usual, we consider the classes of all functions whose modulus of the difference integrates to zero.}

$$\left(\int_{\mathbb{R}} |u(x)|^2 \, dx\right)^{\frac{1}{2}}$$

is finite and of all complex sequences $X = (x_j)_{j \in \mathbb{Z}}$ for which

$$\|X\|_{l^2} = \left(\sum_j |x_j|^2\right)^{\frac{1}{2}}$$

is finite. Let $l^2_k$ represent the set of $k$-vectors with components in $l^2$ and norm

$$\|(X_0, \ldots, X_{k-1})\|_{l^2_k} = \left(\sum_{j=0}^{k-1} \|X_j\|_{l^2}^2\right)^{\frac{1}{2}}.$$

Let $\mathbb{R}$ denote the compactification of the real line $\mathbb{R}$ with the point $\infty$. The set of continuous functions defined on $\mathbb{R}$ (respectively the unit circle $\mathbb{T}$) is represented by $C(\mathbb{R})$ ($C(\mathbb{T})$) and the set of piecewise continuous functions on $\mathbb{R}$ (respectively $\mathbb{T}$), that is, functions with well defined one sided limits at all points of $\mathbb{R}$ ($\mathbb{T}$), by $PC(\mathbb{R})$ ($PC(\mathbb{T})$). These function sets are considered as subalgebras of $L^\infty(\mathbb{R})$ ($L^\infty(\mathbb{T})$), the algebra of essentially bounded measurable functions. If $\Omega$ is a subset of $\mathbb{R}$ we denote by $PC_\Omega(\mathbb{R})$ the subspace of $PC(\mathbb{R})$ of the functions continuous in $\mathbb{R}$, except at the points in $\Omega$. With the notation $a(t^\pm)$ we represent the right/left (resp. clockwise/anticlockwise) one sided limits of the function $a$ at the point $t$ (at the point $\infty$ we use the usual notation $a(\pm \infty)$).

Given a bounded measurable matrix-valued function $a$ acting on the unit circle we define its $k$th (matrix) Fourier coefficient by

$$a_k = \int_0^1 a(e^{2\pi i s}) e^{-2\pi i k s} \, ds. \quad (2.1)$$

The block Laurent operator $T^0(a)$ acting boundly on $l^2_{d+1}$ is then the matrix application with each entry (acting on $l^2$) defined as

$$(T^0(a)^{ij}u^i)_k = \sum_j a_k^j u^i_j \quad 0 \leq i, l \leq d \quad (2.2)$$

Let $\tilde{L}^\infty(\mathbb{R})$ denote the set of all functions $a \in L^\infty(\mathbb{R})$ with well defined limits $a(\pm \infty)$ at infinity. Let $\hat{L}^\infty(\mathbb{R})$ denote the subset of $\tilde{L}^\infty(\mathbb{R})$ such that $a(-\infty) = a(+\infty)$. 
2.2. MULTIPLICATION AND CONVOLUTION OPERATORS

Denote by $F$ the Fourier transform defined on the Schwartz space by

$$(Fu)(y) = \int_{-\infty}^{+\infty} e^{-2\pi i xy} u(x) \, dx, \quad y \in \mathbb{R}$$

and by $F^{-1}$ its inverse

$$(F^{-1}v)(x) = \int_{-\infty}^{+\infty} e^{2\pi i xy} v(y) \, dy, \quad x \in \mathbb{R}.$$ 

The operators $F$ and $F^{-1}$ can be extended continuously to bounded and unitary operators acting in $L^2(\mathbb{R})$ (see for instance [6, Theorem 6.17]) and these extensions will be denoted by the same symbols.

For $a \in PC(\mathbb{R})$ define in $L^2(\mathbb{R})$ the multiplication operator

$$(aI u)(t) = a(t)u(t)$$

and the convolution operator

$$W^0(a) = F^{-1}aF.$$ 

We will call $a$ the generating function (in some works it is also called the symbol or presymbol) of the operators $aI$ or $W^0(a)$. Note that

$$\|aI\| = \|a\|_{L^\infty}$$

and

$$\|W^0(a)\| = \|a\|_{L^\infty}.$$ 

By $\chi_+,-$ are represented, respectively, the operator of multiplication by the characteristic function of the set $\mathbb{R}^+ = [0, +\infty[ \quad (\mathbb{R}^- = ] - \infty, 0[)$, and by $J$ the flip operator $(Ju)(t) = u(-t)$.

Define the singular integral operator on $\mathbb{R}$ as

$$S_{\mathbb{R}}u(x) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{u(y)}{y-x} \, dy.$$ 

The singular integral operator can be written as a convolution operator in the form (see [10] or [22])

$$S_{\mathbb{R}} = W^0(\text{sgn})$$

with sgn denoting the sign function.

The next result is known, but due to its importance and the fact that it is difficult to find a proof specifically dealing with the real line, here it is included one.

**Proposition 2.2.1.** If $f \in C(\mathbb{R})$ then the commutator $S_{\mathbb{R}}f - fS_{\mathbb{R}}$ is compact.
Proof: We begin by introducing the isometric isomorphism

\[ U : L^2(\mathbb{T}) \to L^2(\mathbb{R}), \quad (Uv)(x) = \frac{\sqrt{2}}{i + x} v\left(\frac{i - x}{i + x}\right) \quad (x \in \mathbb{R}) \tag{2.10} \]

and its inverse

\[ U^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{T}), \quad (U^{-1}u)(t) = \frac{i\sqrt{2}}{1 + t} u\left(\frac{1 - t}{1 + t}\right) \quad (t \in \mathbb{T}). \tag{2.11} \]

To prove the result it is sufficient to verify that the operator

\[ U^{-1}(S_{\mathbb{R}} f - f S_{\mathbb{R}})U \]

acting on \( L^2(\mathbb{T}) \) is compact. But

\[ U^{-1}(S_{\mathbb{R}} f - f S_{\mathbb{R}})U = U^{-1} S_{\mathbb{R}} U U^{-1} f U - U^{-1} f U U^{-1} S_{\mathbb{R}} U \tag{2.12} \]

and \( U^{-1} f U \) can be seen to be given by

\[ (U^{-1} f U v)(t) = f\left(\frac{1 - t}{1 + t}\right) v(t). \]

This means that \( U^{-1} f U =: f^# I \) is a multiplication operator in \( L^2(\mathbb{T}) \) with \( f^# \) continuous. Regarding the singular integral operator, \( U^{-1} S_{\mathbb{R}} U \) is given by

\[ (U^{-1} S_{\mathbb{R}} U v)(t) = \frac{i\sqrt{2}}{1 + t} \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{\sqrt{2}}{1 + x} v\left(\frac{1 - s}{1 + s}\right) dx = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{v(s)}{s - t} ds \]

as can be verified by making the change of variables \( x = i\frac{1 - s}{1 + s} \). So the operator in (2.12) can be written as

\[ K = S_{\mathbb{T}} f^# - f^# S_{\mathbb{T}} \]

with \( S_{\mathbb{T}} \) denoting the singular integral operator acting on the unit circle and \( f^# \) continuous. Approaching \( f^# \) uniformly by rational functions \( f^*_n \) without poles on the unit circle we obtain that \( f^*_n(s) - f^#(t) \) is then continuous differentiable in the unit circle and so \( K_n = S_{\mathbb{T}} f^*_n - f^# S_{\mathbb{T}} \) is an integral operator with continuous kernel,

\[ ((S_{\mathbb{T}} f^*_n - f^# S_{\mathbb{T}})v)(t) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{(f^*_n(s) - f^#(t)) v(s)}{s - t} ds, \]

and this operator is compact due to the finite measure of \( \mathbb{T} \) (see, for instance, [18]). As \( \|K - K_n\| \leq 2\|S_{\mathbb{T}}\| \|f^# - f^*_n\| \), we have that the compact operators \( K_n \) approach \( K \) in the norm, and so \( K \) must be compact (see, for instance, [36, Theorem 4.18]).

The proof of the following proposition can be found in [32, Proposition 12.6]. Because that reference is not anymore readily available we decided again to include at least a sketch of the proof here.
Proposition 2.2.2. The following results hold:

(a) If \(a, c \in \dot{L}^\infty(\mathbb{R})\) and \(a(\pm \infty) = c(\pm \infty) = 0\) then \(cW^0(a)\) and \(W^0(a)cI\) are compact operators.

(b) If one of the conditions

(i) \(c \in C(\mathbb{R}), a \in \dot{L}^\infty(\mathbb{R})\)

(ii) \(c \in \dot{L}^\infty(\mathbb{R}), a \in C(\mathbb{R})\)

(iii) \(c \in C(\mathbb{R}) \cap PC(\mathbb{R}), a \in C(\mathbb{R}) \cap PC(\mathbb{R})\)

is fulfilled then the commutator \(cW^0(a) - W^0(a)cI\) is a compact operator.

Proof: To prove the first assertion we start by remarking that \(a\) and \(c\) can be approximated in the \(L^\infty\)-norm by functions with compact support, which means that the operators \(W^0(a)\) and \(cI\) can be approximated in the operator norm by operators with compact support generating functions. So consider \(a\) and \(c\) to have compact support and choose infinitely differentiable functions \(f, g\) such that \(f|_{\text{supp}} = 1\) and \(g|_{\text{supp}} = 1\).

Then

\[cW^0(a) = (cf)W^0(ga) = afW^0(g)W^0(a),\]

and the assertion follows once we have shown that \(fW^0(g)\) is compact. Put \(k = F^{-1}g\). Then for any \(u \in L^2\),

\[(fW^0(g)u)(t) = \int_{-\infty}^{+\infty} f(x)k(x-y)u(y) \, dy,\]

and since \(k\) is an infinitely differentiable function for which \(y^m k(y)\) is bounded for any positive integer \(m\) we have

\[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x)k(x-y)|^2 \, dy \, dx < +\infty\]

and the compactness of \(fW^0(g)\) follows because it is a Hilbert-Schmidt operator (see [12]). It is possible to establish the compactness of \(W^0(a)cI\) in a similar way.

Regarding the second assertion, for the first condition write \(a = a(-\infty)\chi_- + a(+\infty)\chi_+ + a'\) with \(a' \in \dot{L}^\infty(\mathbb{R})\) and \(a'(\pm \infty) = 0\). Then the commutator \(cW^0(a) - W^0(a)cI\) can be written as a sum \(K_1 + K_2 + K_3\) with

\[K_1 = (c - c(\infty))W^0(a'),\quad (2.13)\]
\[K_2 = -W^0(a')(c - c(\infty))I,\quad (2.14)\]
\[K_3 = \frac{a(+\infty) - a(-\infty)}{2} \left( cW^0(\text{sgn}) - W^0(\text{sgn})cI \right).\quad (2.15)\]
The first assertion of this proposition gives immediately the compactness of $K_1$ and $K_2$. The compactness of $K_3$ is verified by noting that $W^0(\text{sgn})$ is the singular integral operator $S_{\mathbb{R}}$ and then applying the commutator property for this operator (see Proposition 2.2.1).

For the second condition write $c = c(-\infty)\chi_- + c(+\infty)\chi_+ + c'$ with $c' \in \dot{L}^\infty(\mathbb{R})$ and $c'(\pm \infty) = 0$. As before, it is possible to write the commutator $cW^0(a) - W^0(a)cI$ as a sum of three operators, $K_1$, $K_2$, $K_3$, with

$$K_1 = c'W^0(a - a(\infty)), \quad (2.16)$$
$$K_2 = -W^0(a - a(\infty))c'I, \quad (2.17)$$
$$K_3 = \frac{c(+\infty) - c(-\infty)}{2} \left(\text{sgn}W^0(a) - W^0(a)\text{sgn}I\right). \quad (2.18)$$

The first two are compact operators by the first assertion of the proposition, and for the third is possible to write

$$\text{sgn}W^0(a) - W^0(a)\text{sgn}I = \text{sgn}F^{-1}aF - F^{-1}aF\text{sgn}I = F^{-1}(F\text{sgn}F^{-1}a - aF\text{sgn}F^{-1})F.$$

As the operator $F\text{sgn}F^{-1}$ is equal to $-S_{\mathbb{R}}$ because $(F^{-1}u)(x) = (Fu)(-x)$ we can apply again the commutator property for the singular integral operator.

Finally for the third condition it is possible to write

$$a(t) = \frac{a(+\infty) + a(-\infty)}{2} + \frac{a(+\infty) - a(-\infty)}{2} \coth \pi \left(t + \frac{i}{2}\right) + a'$$

with $a' \in C(\mathbb{R})$ and $a'(\pm \infty) = 0$. So the commutator $cW^0(a') - W^0(a')cI$ verifies condition (ii) and is compact. We are only left with the commutator $cW^0(\coth \pi(. + \frac{1}{2})) - W^0(\coth \pi(. + \frac{1}{2}))cI$. But by using the Mellin Transform $M$ defined as

$$(Mu)(y) = \int_0^{+\infty} x^{\frac{1}{2}-iy-1}u(x) \, dx, \quad y \in \mathbb{R} \quad (2.19)$$

with inverse

$$(M^{-1}v)(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} x^{iy-1/2}v(y) \, dy, \quad x \in \mathbb{R}^+ \quad (2.20)$$

and the composed transform $E = M^{-1}F$ it is possible to prove that the commutator is equivalent to a commutator between the singular integral operator on the half-axis and a continuous function (see [17, Section 2.1.2]), which is again compact. This can be seen by considering the singular integral operator on the half-line as $\chi_+S_{\mathbb{R}}\chi_+$, using an even extension to the continuous function and applying then Proposition 2.2.1. 


2.3 Auxiliary operators and homomorphisms

Let $\tau$ be a positive real number, $I$ represent the identity operator and define the following operators acting on $L^2(\mathbb{R})$, all with norm 1:

\[
(P_\tau u)(t) = \begin{cases} 
  u(t) & \text{if } |t| < \tau \\
  0 & \text{if } |t| > \tau 
\end{cases}, \quad Q_\tau = I - P_\tau
\]

\[
(R_\tau u)(t) = \begin{cases} 
  u(\tau - t) & \text{if } 0 < t < \tau \\
  u(-\tau - t) & \text{if } -\tau < t < 0 \\
  0 & \text{if } |t| > \tau 
\end{cases}
\]

\[
(R'_\tau u)(t) = \begin{cases} 
  u(2\tau - t) & \text{if } 0 < t < \tau \\
  u(-2\tau - t) & \text{if } -\tau < t < 0 \\
  0 & \text{if } |t| > \tau 
\end{cases}
\]

\[
(V_\tau u)(t) = \begin{cases} 
  0 & \text{if } |t| < \tau \\
  u(t - \tau) & \text{if } t > \tau \\
  u(t + \tau) & \text{if } t < -\tau 
\end{cases}, \quad (V_{-\tau} u)(t) = \begin{cases} 
  u(t + \tau) & \text{if } t > 0 \\
  u(t - \tau) & \text{if } t < 0 
\end{cases}
\]

The following properties of the above defined operators are very important and easily demonstrable:

**Lemma 2.3.1.**

- $R_\tau P_\tau = P_\tau R_\tau = R_\tau$, $P_\tau = R_\tau^2$, $R_\tau^* = R_\tau$, $\|R_\tau\| = 1$;
- $V_\tau V_{-\tau} = Q_\tau$, $V_{-\tau} V_\tau = I$, $(V_\tau)^* = V_{-\tau}$, $\|V_\tau\| = \|V_{-\tau}\| = 1$;
- $J V_{\pm\tau} = V_{\mp\tau} J$, $J R_\tau = R_\tau J$;
- $P_\tau \to I$, $V_{-\tau} \to 0$, $R'_\tau \to 0$.

In order to define the homomorphisms that will identify the local algebras, (see Section 2.4.2 below) we need the following unitary operators, defined in $L^2(\mathbb{R})$:

\[
(Z_\tau u)(x) = \frac{1}{\sqrt{\tau}} u(x/\tau), \text{ with } Z_\tau^{-1} = Z_{\tau^{-1}} \text{ and } \|Z_\tau\| = 1, \quad \tau > 0, \quad (2.21)
\]

\[
(U_s u)(x) = e^{-2\pi i s x} u(x), \text{ with } U_s^{-1} = U_{-s} \text{ and } \|U_s\| = 1, \quad s \in \mathbb{R}, \quad (2.22)
\]

\[
(V_t u)(x) = u(x - t), \text{ with } V_t^{-1} = V_{-t} \text{ and } \|V_t\| = 1, \quad t \in \mathbb{R}. \quad (2.23)
\]

**Proposition 2.3.2.** The following relations hold for $a \in PC(\mathbb{R})$:

- (a) $Z_{\tau}^{-1} a Z_{\tau} \to a(-\infty)\chi_- + a(+\infty)\chi_+$, as $\tau \to +\infty$;
(b) $Z_\tau a Z_\tau^{-1} \rightarrow a(0^-)\chi_+ + a(0^+)\chi_+$, as $\tau \rightarrow +\infty$;

(c) $Z_\tau^{-1} W^0(a) Z_\tau = W^0(Z_\tau a Z_\tau^{-1})$

(d) $U_s F^{-1} = F^{-1} V_s$, $F U_s = V_s F$;

(e) $\tilde{V}_s F^{-1} = F^{-1} \tilde{V}_s$, $F \tilde{V}_s = U_s F$;

Prove: For (a), consider the support of $u \in L^2(\mathbb{R})$ to be contained in the positive half-axis to simplify the notation (this does not imply loss of generality because the operator $\chi_+$ is a multiplication operator and commutes with $Z_\tau$). We have that $(Z_\tau^{-1})^1 a Z_\tau u(x) = a(\tau x)u(x)$. Calculating the norm $\| (a(+\infty)\chi_+ - Z_\tau^{-1} a Z_\tau) u \|$ (squared) we obtain

$$
\int_0^{+\infty} |(a(+\infty) - a(\tau x))u(x)|^2 \, dx.
$$

Given $\epsilon > 0$ choose $x_\epsilon$ such that for $x \geq x_\epsilon$, $|a(+\infty) - a(x)|^2 \|u\|^2 < \frac{\epsilon^2}{2}$. Then for $\tau > 1$ the above integral can be written as the sum

$$
\int_0^{x_\epsilon} |(a(+\infty) - a(\tau x))u(x)|^2 \, dx + \int_{x_\epsilon}^{+\infty} (|a(+\infty) - a(\tau x)||u(x)|)^2 \, dx \leq \max_{0 < x < x_\epsilon} |a(+\infty) - a(x)| \int_0^{x_\epsilon} |u(x)|^2 \, dx + \frac{\epsilon^2}{2},
$$

and for $\tau$ sufficiently large, the first term can be as small as desired. As was said above, it is possible to use the same reasoning if the support of $u$ contained in the negative half-axis, and so the first assertion is proved. The second assertion is proved in the same way. The last three assertions of the proposition are proved by writing the operators explicitly and making an obvious transformation of variables when necessary.

Next we will present a list of weak limits that will be necessary in the main part of this work.

**Proposition 2.3.3.** The following relations hold for $s \in \mathbb{R}$ as $\tau \rightarrow +\infty$.

(a) $Z_\tau^{\pm 1} \rightarrow 0$, $R_\tau \rightarrow 0$, $V_\tau \rightarrow 0$;

(b) $R_\tau U_s Z_\tau^{\pm 1} \rightarrow 0$, $R_\tau \tilde{V}_s Z_\tau^{\pm 1} \rightarrow 0$;

(c) $V_{-\tau} U_s Z_\tau^{\pm 1} \rightarrow 0$, $V_{-\tau} \tilde{V}_s Z_\tau^{\pm 1} \rightarrow 0$;

**Proof:** Let $u = \chi_{[a,b]}$ and $v = \chi_{[c,d]}$ be the characteristic functions of two intervals in $\mathbb{R}$. Then

$$
(v, Z_\tau u) = \int_c^d \frac{1}{\sqrt{\tau}} \chi_{[a,\tau \cdot b]}(x) \, dx \leq \frac{1}{\sqrt{\tau}} (d - c) \rightarrow 0.
$$
So the result is true for any characteristic functions \( u \) and \( v \). But then it is also valid for any piecewise constant functions \( u \) and \( v \) in \( L^2(\mathbb{R}) \) with a finite number of jumps. As these functions are dense in \( L^2(\mathbb{R}) \), by Lemma 2.1.2 the result follows. The same reasoning applies to all the other cases, so it is enough to verify the convergence to zero only with characteristic functions. So we have

\[
(v, Z^{-1}_\tau u) = \int_c^d \sqrt{\tau} \chi_{[\frac{a}{\tau}, \frac{b}{\tau}]}(x) \, dx \leq \sqrt{\tau} (\frac{b}{\tau} - \frac{a}{\tau}) \to 0.
\]

It is equally easy to see the convergence for the operators \( R_\tau \) and \( V_\tau \). For the other cases denote by \( B_\tau \) any of the operators \( R_\tau \tilde{V}_s \), \( V_{-\tau} \tilde{V}_s \), \( R_\tau U_s \) and \( V_{-\tau} U_s \). We have then that

\[
|(B_\tau \chi_{[a, b]})(x)| \leq 1
\]

\[
|(B^*_\tau \chi_{[c, d]})(x)| \leq 1
\]

for any possible \( a, b, c, d, \tau, s \) and \( x \). This means that

\[
\left| (\chi_{[c, d]}, B_\tau Z_\tau \chi_{[a, b]}) \right| = \left| \frac{1}{\sqrt{\tau}} (\chi_{[c, d]}, B_\tau \chi_{[a, b]}) \right| =
\]

\[
= \frac{1}{\sqrt{\tau}} \int_c^d (B_\tau \chi_{[a, b]})(x) \, dx \leq \frac{1}{\sqrt{\tau}} (d - c) \to 0
\]

and

\[
\left| (\chi_{[c, d]}, B_\tau Z^{-1}_\tau \chi_{[a, b]}) \right| = \left| \sqrt{\tau} (B^*_\tau \chi_{[c, d]}, \chi_{[\frac{a}{\tau}, \frac{b}{\tau}]}) \right| =
\]

\[
= \sqrt{\tau} \int_{\frac{a}{\tau}}^{\frac{b}{\tau}} (B^*_\tau \chi_{[c, d]})(x) \, dx \leq \sqrt{\tau} (\frac{b}{\tau} - \frac{a}{\tau}) \to 0,
\]

which proves (b) and (c).

\[\blacksquare\]

### 2.4 Algebraic techniques

#### 2.4.1 \( C^* \)-algebras properties

Due to the importance of the properties exhibited by \( C^* \)-algebras in this work, we will enunciate here some of the non-trivial ones (see, for instance, [8] or [26]).

(a) If \( A \) is a \( C^* \)-algebra and \( \mathcal{J} \) is a closed two-sided ideal of \( A \) then \( \mathcal{J}^* = \mathcal{J} \) and \( A/\mathcal{J} \) provided with the involution \((a + \mathcal{J})^* = a^* + \mathcal{J} \) is a \( C^* \)-algebra.
(b) If \( A \) is a \( C^* \)-algebra, \( B \) is a \( C^* \)-subalgebra of \( A \), and \( J \) is a closed two-sided ideal of \( A \) then \( B + J \) is a \( C^* \)-subalgebra of \( A \), and the \( C^* \)-algebras \( (B + J)/J \) and \( B/(B \cap J) \) are isometric isomorphic.

(c) If \( A \) is a unital \( C^* \)-algebra and \( B \) is a \( C^* \)-subalgebra of \( A \) containing the identity, then if an element \( a \in B \) is invertible in \( A \), the inverse is also in \( B \) (i.e., \( B \) is inverse closed in \( A \)).

(d) \( C^* \)-algebras are semi-simple.

2.4.2 Localization

We will use the Local Principle of Allan, which is a generalization of the Gelfand theory to non-commutative Banach algebras, but that are close to commutative ones by having a rich center. We will only briefly describe the principle, applied to our \( C^* \) case. For more detailed information we recommend the reader to [17, Chapter 1]. Let \( \mathcal{X} \) be a unital \( C^* \)-algebra and \( \mathcal{Y} \) be a closed unital \( C^* \)-subalgebra of the center of \( \mathcal{X} \). Then \( \mathcal{Y} \) is a commutative algebra and we denote its maximal ideal space by \( M(\mathcal{Y}) \). To each element \( x \) of \( M(\mathcal{Y}) \) we associate the smallest closed two-sided ideal \( I_x \) of \( \mathcal{X} \) which contains \( x \). By \( \Phi_x \) we denote the canonical homomorphism from \( \mathcal{X} \) onto the quotient algebra \( \mathcal{X}/I_x \). Then the Local Principle of Allan affirms that an element \( a \in \mathcal{X} \) is invertible if and only if the cosets \( \Phi_x(a) \) are invertible in the quotient (local) algebras \( \mathcal{X}/I_x \) for every \( x \in M(\mathcal{Y}) \), and that the mapping \( M(\mathcal{Y}) \to \mathbb{R}^+, x \mapsto \|\Phi_x(a)\| \) is upper semi-continuous for each \( a \in \mathcal{A} \).

2.4.3 Flip elimination technique

Our main tool to treat algebras which contain a flip (that is, an element that squared gives the identity) with certain properties, is to eliminate flip by doubling the dimension. Some ideas regarding this technique were presented in [32]. The concrete flip elimination scheme was presented in [33, Scheme 3.3].

Lemma 2.4.1. Let \( \mathcal{X} \) be a \( C^* \)-algebra with identity \( e \) whose center contains a selfadjoint projection \( p \). Let \( \mathcal{Y} \) be the algebra generated by \( \mathcal{X} \) and a selfadjoint flip \( j \) with the properties \( j\mathcal{X}j \subset \mathcal{X} \) and in particular \( jpj = e - p \). Then any element \( a \) of \( \mathcal{Y} \) can be written uniquely as the sum \( a_1 + a_2j \) with \( a_{1,2} \in \mathcal{X} \) and the mapping \( L : \mathcal{Y} \to [p\mathcal{X}p]^{2 \times 2}, \)

\[
a \mapsto \begin{bmatrix} pa_1 p & pa_2 p \\ p\tilde{a}_2 p & p\tilde{a}_1 p \end{bmatrix}
\]

with \( \tilde{a} = ja j \) is an isometric \(*\)-isomorphism.

Proof: In [33, Scheme 3.3] it was already showed that \( L \) is an isometric continuous homomorphic embedding. We have only to prove it is onto. Let \( a_i, \ i = 1 \ldots 4 \) be any
four elements of $\mathcal{X}$. If we define $a = pa_1 p + pa_2 j (e - p) + (e - p) j a_3 p + (e - p) \tilde{a}_4 (e - p)$, then $a \in \mathcal{Y}$ and it easy to see that

$$L(a) = \begin{bmatrix} pa_1 p & pa_2 p \\ pa_3 p & pa_4 p \end{bmatrix},$$

which proves that $L$ is onto.
Chapter 3

The finite section method

3.1 Introduction

In terms of practical applications, it may seem to be a waste of time to study the pure finite section method, because the resulting approximating functions are still in infinite dimensional spaces and are not simple as for example polynomials, which percludes their easy treatment with the help of computers. Nevertheless, due to its mathematical simplicity (compared with spline methods) on one hand, and its non-triviality on the other hand, so that one can develop new techniques and obtain insights as how to solve the more complex problems with spline approximation methods, we opted to include this chapter where the ideas and results are presented in a more detailed form. We will now introduce the finite section approximation method, and then use the procedure of algebraization-essentialization-localization-identification described in the introduction.

Consider the approximation method (see Section 2.3)

\[(P_\tau A P_\tau + Q_\tau)u_\tau = v, \quad \tau > 0\] (3.1)

to solve the equation

\[Au = v \quad A \in \mathcal{L}(L^2(\mathbb{R})) \quad u, v \in L^2(\mathbb{R}),\]

with \(u_\tau \in L^2(\mathbb{R})\). This method is called the finite section method.

Definition 3.1.1. We say that the finite section method (3.1) applies to A if and only if

(a) there exists a positive constant \(\tau_0\), such that for any \(\tau > \tau_0\) and any \(v\) in \(L^2(\mathbb{R})\) there exists a unique solution \(u_\tau\) of the equation \((P_\tau A P_\tau + Q_\tau)u_\tau = v;\)

(b) the sequence \((u_\tau)\) converges to a solution of the equation \(Au = v\) as \(\tau \to +\infty\).
3.2 Algebraization

Let $E$ be the set formed by all the sequences of operators $(A_\tau)_{\tau \in \mathbb{R}^+}$, $A_\tau : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, such that $\sup_\tau \|A_\tau\|_{\mathcal{L}(L^2)} < \infty$. This set with the operations $(A_\tau) + (B_\tau) = (A_\tau + B_\tau)$ and $(A_\tau)(B_\tau) = (A_\tau B_\tau)$, with the norm $\|(A_\tau)\| = \sup_\tau \|A_\tau\|_{\mathcal{L}(L^2)}$ and involution $(A_\tau)^* = (A_\tau^2)$ is a unital $C^*$-algebra. Note that the constant sequences $(A_\tau)$ are included in $E$ for any $A \in \mathcal{L}(L^2)$. Let now $G$ be the closed two-sided ideal in $E$ of all sequences $(A_\tau)$ for which we have $\lim_{\tau \to \infty} \|A_\tau\| = 0$.

Definition 3.2.1. We say that the sequence $(A_\tau) \in E$ is stable if there exists a positive constant $\tau_0$, such that for any $\tau > \tau_0$, $A_\tau$ is invertible and $\sup_{\tau > \tau_0} \|A_\tau^{-1}\| < \infty$.

The next result is then well known, but for the readers convenience we include a proof here:

Theorem 3.2.1. The following propositions are equivalent for $A_\tau = P_\tau A \tau + Q_\tau$, $A \in \mathcal{L}(L^2(\mathbb{R}))$

(a) The approximation method \[3.1\] applies to $A$;

(b) $(A_\tau)$ is stable and $A$ is invertible;

(c) $A$ is invertible and the coset $(A_\tau) + G$ is invertible in the quotient algebra $E/G$.

Proof: (a) $\Rightarrow$ (b): (a) implies that there exists a positive constant $\tau_0$ such that for any $\tau > \tau_0$, $A_\tau$ is invertible and $A_\tau^{-1}v$ converges for any $v \in L^2$. By the Banach-Steinhaus Theorem $\sup_{\tau > \tau_0} \|A_\tau^{-1}\| < \infty$. To prove that $A$ is invertible note that

$$\|u - A_\tau^{-1}Au\| \leq \|A_\tau^{-1}\|\|A_\tau u - Au\|$$

the first factor of the right-hand side being uniformly bounded and the second going to zero as $\tau$ increases. So if $u \in \ker A$ we have immediately that $\|u\| = 0$ and this means that $u = 0$. So ker $A = \{0\}$, and as $\text{Im}\{A\} = L^2$ (by (a)), $A$ is invertible.

(b) $\Rightarrow$ (c): Put $B_\tau = A_\tau^{-1}$ for $\tau > \tau_0$ and $B_\tau = I$ in the other cases. Then it is not difficult to see that

$$(B_\tau)(A_\tau) = (A_\tau)(B_\tau) \in (I + G).$$

(c) $\Rightarrow$ (a): Let $(B_\tau) + G$ be the inverse of $(A_\tau) + G$. Then $B_\tau A_\tau = I + C_\tau$, $A_\tau B_\tau = I + D_\tau$, with $C_\tau$, $D_\tau \in G$. Choose $\tau_0$ such that $\|C_\tau\| < 1/2$, $\|D_\tau\| < 1/2$ for $\tau > \tau_0$. Then $I + C_\tau$ and $I + D_\tau$ are invertible for $\tau > \tau_0$ and $\sup_{\tau > \tau_0} \|(I + C_\tau)^{-1}\| \leq 2$. The operators $A_\tau$ are thus two-sided invertible and this permits us to write

$$A_\tau u_\tau = v \Leftrightarrow B_\tau A_\tau u_\tau = B_\tau v \Leftrightarrow u_\tau = (I + C_\tau)^{-1}B_\tau v.$$

We only have to prove now that $\|u - u_\tau\|_{L^2} \to 0$ with $u = A^{-1}v$. This follows from

$$\|u - u_\tau\| = \|u - (I + C_\tau)^{-1}B_\tau Au\| \leq \|(I + C_\tau)^{-1}B_\tau\| \|A_\tau u - Au\|,$$

the last term going to zero because of the strong convergence $\lim A_\tau = A$, and the uniform boundedness of $B_\tau$.

$\blacksquare$
3.3 Essentialization

Of course, the algebra $E$ is too general to derive any results regarding our problems. For example, the sequences don’t even have to be strongly convergent. In order to be able to work, we introduce the following restrictions. Let $F \subset E$ be the set of all sequences $(A_\tau)$ for which there exist operators $A$, $A_{ij}$, $i, j = 1, 2$ such that the following strong limits as $\tau \to \infty$ exist:

- $A_\tau \to A$ and $A^*_\tau \to A^*$;
- $R_\tau A_\tau R_\tau \to A_{11}$ and $(R_\tau A_\tau R_\tau)^* \to A_{11}^*$;
- $R_\tau A_\tau V_\tau \to A_{12}$ and $(R_\tau A_\tau V_\tau)^* \to A_{12}^*$;
- $V_{-\tau} A_\tau R_\tau \to A_{21}$ and $(V_{-\tau} A_\tau R_\tau)^* \to A_{21}^*$;
- $V_{-\tau} A_\tau V_\tau \to A_{22}$ and $(V_{-\tau} A_\tau V_\tau)^* \to A_{22}^*$.

It should be remarked that the operators $A_{ij}$, $i, j = 1, 2$ don’t depend in principle only on the operator $A$, but more on the sequence $(A_\tau)$. Using the fact that $I = R_\tau R_\tau + V_\tau V_{-\tau}$ and Lemma 2.3.1, one can see that this set is actually a closed $C^*$-subalgebra of $E$ which contains $G$.

Now let $K \subset \mathcal{L}(L^2(\mathbb{R}))$ denote the ideal of compact operators and define $J_0$ and $J_1$ to be the sets

\[ J_0 = \{(K) + (G_\tau), \ K \in K, \ (G_\tau) \in G\}, \]
\[ J_1 = \{ (R_\tau K_1 R_\tau + R_\tau K_2 V_{-\tau} + V_\tau K_3 R_\tau + V_\tau K_4 V_{-\tau}) + (G_\tau), \ K_k \in K, \ (G_\tau) \in G\}. \]

(3.2)

Proposition 3.3.1. $J_0$ is a closed two sided ideal of $F$.

Proof: As it is clear that $J_0$ is a linear subspace of $F$, to prove that it is a left ideal it is enough to see that the sequences $(A_\tau)(K)$ are in $J_0$. Writing

\[ (A_\tau)(K) = (AK) - ((A - A_\tau)K) \]

one can see that the first term of the right-hand side is obviously in $J_0$, and the second is even in $G$ because $A_\tau \to A$ strongly as $\tau \to \infty$. To prove that $J_0$ is a right ideal note that

\[ (K)(A_\tau) = (KA) - (K(A - A_\tau)) \]

and $A^*_\tau \to A^*$. To prove that it is closed, remember first that if $(j_\tau) = (K + G_\tau)$, then as $j_\tau$ converges to $K$ in the norm, we have $\|K\| = \lim_{\tau \to \infty} \|K + G_\tau\|$. Then consider a Cauchy sequence in $J_0$, $(j_\tau)^{(k)} = (K^{(k)}) + G_\tau^{(k)}$ and because of the above result the sequence $K^{(k)}$ is also a Cauchy sequence, which means that there exists a compact operator $K$ such that $\|K - K^{(k)}\| \to 0$. But then the sequence $(G_\tau^{(k)})$ is also Cauchy,
and there exists \((G_\tau)\) such that \(\|G_\tau - G_\tau^{(k)}\| \to 0\). We conclude that the sequence \((j_\tau) = (K + G_\tau)\) is the limit of \((j_\tau)^{(k)}\) and this ends the proof. 

**Proposition 3.3.2.** \(\mathcal{J}_1\) is a closed two sided ideal of \(\mathcal{F}\). 

**Proof:** We will only prove that \(\mathcal{J}_1\) is a left ideal. The proof that it is a right ideal is similar. For \((A_\tau) \in \mathcal{F}\), 

\[
A_\tau R_\tau K_1 R_\tau = R_\tau R_\tau A_\tau R_\tau K_1 R_\tau + V_\tau V_\tau A_\tau R_\tau K_1 R_\tau = \\
R_\tau A_{11} K_1 R_\tau + R_\tau (R_\tau A_\tau R_\tau - A_{11}) K_1 R_\tau + V_\tau A_{21} K_1 R_\tau + V_\tau (V_\tau A_\tau R_\tau A_{21}) K_1 R_\tau,
\]

the sequence corresponding to the second and fourth terms being in the ideal \(\mathcal{G}\) due to the strong limits and the presence of the compact operator \(K_1\). In a similar way we have 

\[
A_\tau R_\tau K_2 V_\tau = R_\tau A_{11} K_2 V_\tau + V_\tau A_{21} K_2 V_\tau + G_\tau^{(1)}, \\
A_\tau V_\tau K_3 R_\tau = R_\tau A_{12} K_3 R_\tau + V_\tau A_{22} K_3 R_\tau + G_\tau^{(2)}
\]

and 

\[
A_\tau V_\tau K_4 V_\tau = R_\tau A_{12} K_4 V_\tau + V_\tau A_{22} K_4 V_\tau + G_\tau^{(3)}
\]

with \((G_\tau^{(k)}) \in \mathcal{G}\) and this proves that \(\mathcal{J}_1\) is a left ideal. The closedness proof is as in Proposition 3.3.1, taking into account the existence of the strong limits of the algebra definition, in order to apply the Banach-Steinhaus Theorem.

We can now define the \(^*\)-homomorphisms in \(\mathcal{F}\)

\[
O^0 : \mathcal{F} \to \mathcal{L}(L^2), \quad O^0((A_\tau)) = A \tag{3.3}
\]

\[
O^1 : \mathcal{F} \to \mathcal{L}(L^2)^{2 \times 2}, \quad O^1((A_\tau)) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3.4}
\]

It is not difficult to see that \(O^i(\mathcal{G}) = 0 \ (i \in \{0, 1\})\), that 

\[
O^0((K)) = K, \quad O^0((R_\tau K_1 R_\tau + R_\tau K_2 V_\tau + V_\tau K_3 R_\tau + V_\tau K_4 V_\tau)) = 0,
\]

and 

\[
O^1((K)) = 0, \quad O^1((R_\tau K_1 R_\tau + R_\tau K_2 V_\tau + V_\tau K_3 R_\tau + V_\tau K_4 V_\tau)) = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}
\]

for every \(K, K_k \in \mathcal{K}\).

The sets in \(\mathcal{F}/\mathcal{G}\), \(\{j_0 + \mathcal{G}, \ j_0 \in \mathcal{J}_0\}\) and \(\{j_1 + \mathcal{G}, \ j_1 \in \mathcal{J}_1\}\) are naturally closed two sided ideals. Let \(\mathcal{J}\) denote the smallest closed two-sided ideal in \(\mathcal{F}\) containing both \(\mathcal{J}_0\) and \(\mathcal{J}_1\).

We are now in the conditions to apply the Lifting Theorem \([17, \text{Theorem 1.8}]\) and obtain the result:
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**Theorem 3.3.3.** The coset \((A_\tau) + \mathcal{G}\) is invertible in the quotient algebra \(\mathcal{F}/\mathcal{G}\) if and only if the operators \(O^0((A_\tau))\), \(O^1((A_\tau))\) are invertible in \(\mathcal{L}(L^2)\) (resp. \(\mathcal{L}(L^2)^{2\times2}\)) and the coset \((A_\tau) + \mathcal{J}\) is invertible in \(\mathcal{F}\mathcal{J} := \mathcal{F}/\mathcal{J}\).

If the sequence \(A_\tau\) is of the form \(P_\tau A P_\tau + Q_\tau\) then

\[
O^1(A_\tau) = \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix}
\]

and so the last theorem gives,

**Corollary 3.3.4.** The method (3.1) applies to \(A\) if and only if \(A\) and \(A_{11}\) are invertible, and \((A_\tau) + \mathcal{J}\) is invertible in \(\mathcal{F}\mathcal{J}\).

Due to this last result and the fact that in the previous literature the strong limit \(A_{11}\) is usually represented by \(\tilde{A}\), we will use in the future \(\tilde{A}\) instead of \(A_{11}\).

Having the above results regarding the algebra \(\mathcal{F}\), it must now be verified that this algebra actually contains the sequences that interest us. For that we have the next propositions. The proofs are either immediate or very similar to one another. The author decided to present only the proofs that need some new idea. Also there will be made extensive use of the properties and relations described in Lemma 2.3.1 without reference.

**Proposition 3.3.5.** The following relations hold:

\[
R_\tau P_\tau R_\tau \to I \quad R_\tau P_\tau V_\tau = 0 \quad V_{-\tau} P_\tau R_\tau = 0 \quad V_{-\tau} P_\tau V_\tau = 0 \quad (3.5)
\]

**Proof:** The proof of these results is immediate.

**Proposition 3.3.6.** The following relations are true for \(c \in PC(\mathbb{R})\):

\[
R_\tau cI R_\tau \to c(-\infty)\chi_+ + c(+\infty)\chi_+; \quad (3.6) \\
R_\tau cI V_\tau = 0; \quad (3.7) \\
V_{-\tau} cI R_\tau = 0; \quad (3.8) \\
V_{-\tau} cI V_\tau \to c(-\infty)\chi_+ + c(+\infty)\chi_+. \quad (3.9)
\]

**Proof:** We have

\[
(R_\tau cI R_\tau u)(x) = \begin{cases} 
  c(\tau - x)u(x) & \text{if } 0 < x < \tau \\
  c(-\tau - x)u(x) & \text{if } -\tau < x < 0 \\
  0 & \text{if } |x| > \tau
\end{cases}
\]
For a more simple notation suppose that supp$(c) \subset \mathbb{R}^+$. We must then prove that $\|(R_{\tau}cI R_{\tau} - c(+\infty)\chi_+)u\| \to 0$ for any fixed $u \in L^2$. Let $m = \max_x |c(x) - c(+\infty)|$.

For any $\epsilon > 0$ there exists $\tau_1$ such that

$$\tau \geq \tau_1 \Rightarrow |c(\tau) - c(+\infty)| < \epsilon,$$

$$\tau \geq \tau_1 \Rightarrow \int_{\tau}^{+\infty} |u(x)|^2 \, dx < \epsilon.$$

If $\tau_0 = 2\tau_1$ we have then for $\tau \geq \tau_0$,

$$\|(R_{\tau}cI R_{\tau} - c(+\infty)\chi_+)u\|^2 =$$

$$\int_0^{\tau} |(c(\tau - x) - c(+\infty))u(x)|^2 \, dx + \int_{\tau}^{+\infty} |c(+\infty)u(x)|^2 \, dx =$$

$$\int_0^{\tau-\tau_1} |(c(\tau - x) - c(+\infty))u(x)|^2 \, dx + \int_{\tau-\tau_1}^{\tau} |(c(\tau - x) - c(+\infty))u(x)|^2 \, dx +$$

$$\int_{\tau}^{+\infty} |c(+\infty)u(x)|^2 \, dx \leq \epsilon^2 \int_0^{\tau-\tau_1} |u(x)|^2 \, dx + (m^2 + c(+\infty)^2)\epsilon,$$

which ends the proof of the first assertion.

The second and third assertions are immediate. For the last one we have only to note that

$$(V_{-\tau}cI V_{\tau}u)(x) = \begin{cases} c(x + \tau)u(x) & \text{if } x > 0 \\ c(x - \tau)u(x) & \text{if } x < 0 \end{cases}$$

and use a reasoning similar to the one above.

For a function $a \in PC(\mathbb{R})$ let $\hat{a}(x) = a(-x)$. Then

**Proposition 3.3.7.** The following relations hold for $a \in PC(\mathbb{R})$:

\begin{align*}
R_{\tau}\chi_+ W^0(a)\chi_+ R_{\tau} &= P_{\tau}\chi_+ W^0(\hat{a})\chi_+ P_{\tau} \quad (3.10) \\
R_{\tau}\chi_- W^0(a)\chi_- R_{\tau} &= P_{\tau}\chi_- W^0(\hat{a})\chi_- P_{\tau} \quad (3.11) \\
R_{\tau}\chi_+ W^0(a)\chi_- R_{\tau} &\to 0 \quad (3.12) \\
R_{\tau}\chi_- W^0(a)\chi_+ R_{\tau} &\to 0 \quad (3.13)
\end{align*}

**Proof:** The first two assertions are easily proved by writing the operators explicitly. The third and fourth are similar, and we will only prove the third. If $x < 0$ or $x > \tau$ the function $R_{\tau}\chi_+ W^0(a)\chi_- R_{\tau}u$ gives the value 0. For $0 < x < \tau$ we have

$$(R_{\tau}\chi_+ W^0(a)\chi_- R_{\tau}u)(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(\tau-x)\xi} a(\xi) \int_{-\tau}^{0} e^{i\xi y} u(-\tau - y) \, dy \, d\xi =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i((2\tau-x)-\xi) y} a(\xi) \int_{-\tau}^{0} e^{i\xi y'} u(y') \, dy' \, d\xi,$$
and this means
\[ R_\tau \chi_a W_0(a) \chi_{-} R_\tau = R'_\tau J_\chi W_0(\tilde{a}) \chi_{-} P_\tau \]
which converges strongly to zero

The proofs of the following Propositions are now similar to the above one.

**Proposition 3.3.8.** The following relations hold for \( a \in PC(\mathbb{R}) \):
\[
\begin{align*}
R_\tau \chi_a W_0(a) \chi_{+} V_\tau &= P_\tau \chi_a JW_0(a) \chi_{+} \quad (3.14) \\
R_\tau \chi_a W_0(a) \chi_{-} V_\tau &= P_\tau \chi_a JW_0(a) \chi_{-} \quad (3.15) \\
R_\tau \chi_a W_0(a) \chi_{-} V_\tau &\to 0 \quad (3.16) \\
R_\tau \chi_a W_0(a) \chi_{+} V_\tau &\to 0 \quad (3.17)
\end{align*}
\]

**Proposition 3.3.9.** The following relations hold for \( a \in PC(\mathbb{R}) \):
\[
\begin{align*}
V_- \tau \chi_a W_0(a) \chi_{+} R_\tau &= \chi_a W_0(a) J_\chi P_\tau \quad (3.18) \\
V_- \tau \chi_a W_0(a) \chi_{-} R_\tau &= \chi_a W_0(a) J_\chi P_\tau \quad (3.19) \\
V_- \tau \chi_a W_0(a) \chi_{-} R_\tau &\to 0 \quad (3.20) \\
V_- \tau \chi_a W_0(a) \chi_{+} R_\tau &\to 0 \quad (3.21)
\end{align*}
\]

**Proposition 3.3.10.** The following relations hold for \( a \in PC(\mathbb{R}) \):
\[
\begin{align*}
V_- \tau \chi_a W_0(a) \chi_{+} V_\tau &= \chi_a W_0(a) \chi_{+} \quad (3.22) \\
V_- \tau \chi_a W_0(a) \chi_{-} V_\tau &= \chi_a W_0(a) \chi_{-} \quad (3.23) \\
V_- \tau \chi_a W_0(a) \chi_{-} V_\tau &\to 0 \quad (3.24) \\
V_- \tau \chi_a W_0(a) \chi_{+} V_\tau &\to 0 \quad (3.25)
\end{align*}
\]

Now we are able to define subalgebras of \( F \) that contain the sequences we are interested in. We will first study an algebra without flip, and after that we will make a generalization to an algebra containing the flip. The main reason for this is that the results for the algebra without the flip are necessary when trying to solve the problem for the algebra with the flip.

### 3.4 Finite section in algebras generated by multiplication and convolution operators

Let \( A \) be the \( C^* \)-subalgebra of \( F \) generated by the constant sequences \((cI), (W_0(a))\) with \( a, c \in PC(\mathbb{R}) \) and the sequence \((P_\tau)\). Designate by \( A/\mathcal{J} \) the quotient algebra
\[
\frac{A}{A \cap \mathcal{J}} \cong \frac{A + \mathcal{J}}{\mathcal{J}}. \quad (3.26)
\]

This algebra has a rich center, and we will make use of it through localization in order to obtain invertibility criteria for the elements.
3.4.1 First localization

We will start by finding a central subalgebra of $\mathcal{A}^\mathcal{J}$.

Proposition 3.4.1. The cosets $(fI) + \mathcal{J}$ with $f \in C(\hat{\mathbb{R}})$ belong to the center of $\mathcal{A}^\mathcal{J}$.

Proof: As we have immediately that $P_\tau fI = fP_\tau$ and $cfI = fcI$, we are only left with the commutator involving the convolution operators $W^0(a)$. As in Proposition 2.2.2 is proved that $fW^0(a) - W^0(a)fI$ is a compact operator the result follows. ■

This last result means that we can apply the Local Principle of Allan (see above Section 2.4.2 or for instance [17] Theorem 1.5]) and localize $\mathcal{A}^\mathcal{J}$ over the central subalgebra $\mathcal{C}$ generated by the set of cosets $\{(fI) + \mathcal{J}, f \in C(\hat{\mathbb{R}})\}$. The maximal ideal space of this subalgebra is isomorphic to $\mathbb{R}$, with a maximal ideal consisting of the cosets $\{(f_xI) + \mathcal{J}, f_x \in C(\hat{\mathbb{R}}), f(x) = 0\}$ (these ideals are indeed not trivial, as one can see using the homomorphisms $O_x$ defined below). Let $\mathcal{I}_x$ denote the smallest closed two-sided ideal in $\mathcal{A}^\mathcal{J}$ containing the ideal $x$ of $\mathcal{C}$. The result of this localization is that the invertibility problem in $\mathcal{A}^\mathcal{J}$ is transferred to an invertibility problem in each of the local algebras $\mathcal{A}^\mathcal{J}_x := \mathcal{A}^\mathcal{J}/\mathcal{I}_x$. Let $\Phi^\mathcal{J}_x$ denote the (canonical) homomorphism from $\mathcal{A}$ to $\mathcal{A}^\mathcal{J}_x$. ■

Lemma 3.4.2. If $c \in PC(\hat{\mathbb{R}})$ such that $c$ is continuous at $x$ and $c(x) = 0$, then $\Phi^\mathcal{J}_x(cI) = 0$

Proof: For $x \neq \infty$ let $f_\epsilon \in C(\hat{\mathbb{R}})$ such that $0 \leq f_\epsilon < 1$ except at $x$, where $f_\epsilon(x) = 1$, and the support of $f_\epsilon$ is contained in the interval $[x - \epsilon, x + \epsilon]$. We have that $\Phi^\mathcal{J}_x(f_\epsilon I)$ is the identity and then

$$\|\Phi^\mathcal{J}_x(cI)\| = \|\Phi^\mathcal{J}_x(cI)\Phi^\mathcal{J}_x(f_\epsilon I)\| = \|\Phi^\mathcal{J}_x(cf_\epsilon I)\| \leq \|cf_\epsilon\|_{L^\infty}$$

and this last norm can be as small as desired by choosing $\epsilon$ small enough. For $x = \infty$ the proof is similar, with the support of $f_\epsilon$ contained in $\{y \in \mathbb{R} : |y| > 1/\epsilon\}$. ■

Proposition 3.4.3. Let $(A_\tau) \in \mathcal{A}$. For $x \neq \infty$ we have $\Phi^\mathcal{J}_x(A_\tau) = \Phi^\mathcal{J}_x(O^0((A_\tau)))$.

Proof: The assertion for the constant sequences $\Phi^\mathcal{J}_x(A)$ is obvious. For $\Phi^\mathcal{J}_x(P_\tau)$, let $y$ be greater then $|x|$ and define $f_y = f_\epsilon$ to be a continuous function supported in the interval $]-y, y[$. Then as $\Phi^\mathcal{J}_x(f_y I)$ is the identity in the local algebra we have $\Phi^\mathcal{J}_x(Q_\tau) = \Phi^\mathcal{J}_x(f_y I)\Phi^\mathcal{J}_x(Q_\tau) = \Phi^\mathcal{J}_x(f_y Q_\tau)$. As $\|f_y Q_\tau\| \to 0$, we conclude that $\Phi^\mathcal{J}_x(Q_\tau) = 0$, which means that $\Phi^\mathcal{J}_x(P_\tau) = \Phi^\mathcal{J}_x(1 - Q_\tau) = I - \Phi^\mathcal{J}_x(Q_\tau) = I$. ■

\footnote{In order not to burden the notation, for $(A_\tau) \in \mathcal{A}$ we write $\Phi^\mathcal{J}_x(A_\tau)$ instead of $\Phi^\mathcal{J}_x((A_\tau))$.}
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Note that the above result means that the projection $P_{\tau}$ does not appear explicitly in the local algebras, and we can treat the local sequences as constant sequences. For example we have

$$\Phi_x^{\tau}(P_{\tau}ABP_{\tau}CP_{\tau}) = \Phi_x^{\tau}(P_{\tau})\Phi_x^{\tau}(AB)\Phi_x^{\tau}(P_{\tau})\Phi_x^{\tau}(C)\Phi_x^{\tau}(P_{\tau}) = \Phi_x^{\tau}(ABC).$$

For a description of the Fredholm properties of the elements of this and other algebras see [11] or [32]. The next result gives us a sufficient condition for invertibility in the local algebras, and because a little reasoning shows that condition to be also necessary for the invertibility in the larger algebra, it is possible to skip the identification process here.

Corollary 3.4.4. If $O^0((A_{\tau}))$ is a Fredholm operator, then $\Phi_x^{\tau}(A_{\tau})$ is invertible in $A_x^{\tau}$ for all $x \in \mathbb{R}$.

**Proof:** If $A = O^0((A_{\tau}))$ is Fredholm, there exists an operator $B$ (with $(B) \in A$) and compact operators $K_{1,2}$ such that $AB = I + K_1$, $BA = I + K_2$. But then $\Phi_x^{\tau}(A_{\tau})\Phi_x^{\tau}(B) = \Phi_x^{\tau}(I + K_1) = I$. Also $\Phi_x^{\tau}(B)\Phi_x^{\tau}(A_{\tau}) = I$ and we proved that $\Phi_x^{\tau}(A_{\tau})$ is invertible.

In the case $x = \infty$, the algebra $A_x^{\tau}$ is still too large for a positive identification. But as this algebra still has a rich center, we can localize again.

### 3.4.2 Second localization

Before we proceed with the second localization, we will define a homomorphism, that will be later seen to be a symbol homomorphism that give us locally equivalent representations (see [17]).

For that, given a sequence $(A_{\tau}) \in A$, we can define a transformation $S_{yr}$ such that to the operator $A_{\tau}$ associates the operator

$$S_{yr}(A_{\tau}) = Z_{\tau}^{-1}U_yA_{\tau}U_{-y}Z_{\tau}, \; y \in \mathbb{R}. \quad (3.27)$$

Note that $S_{yr}$ is multiplicative, i.e. $S_{yr}(A_{\tau}B_{\tau}) = S_{yr}(A_{\tau})S_{yr}(B_{\tau})$, and $\|S_{yr}(A_{\tau})\|_{\mathcal{L}(L^2)} \leq \|A_{\tau}\|_{\mathcal{L}(L^2)}$. Defining now, if it exists, the strong limit

$$S_y(A_{\tau}) = s\text{-lim}_{\tau \to \infty}S_{yr}(A_{\tau}), \quad (3.28)$$

we obtain the next lemma.

**Lemma 3.4.5.** If $(A_{\tau}) \in A$, then the limit $S_y(A_{\tau})$ exists. In particular,

(a) $S_y(P_{\tau}) = P_1$;

(b) $S_y(cI) = c(-\infty)\chi_- + c(+\infty)\chi_+$ for $c \in PC(\mathbb{R})$;
(c) \( S_y(W^0(a)) = a(y^-)W^0(\chi_-) + a(y^+)W^0(\chi_+) \) for \( a \in PC(\hat{\mathbb{R}}) \);

(d) If \( (j_\tau) \in \mathcal{J}_0 \) or \( (j_\tau) \in \mathcal{J}_1 \), then \( S_y(j_\tau) = 0 \).

**Proof:** The assertion (a) is proved by a simple calculation. Assertions (b) and (c) are proved using Proposition [2.3.2]. To prove the last one note that for \( G_\tau \in \mathcal{G} \), \( \|S_\chi(G_\tau)\| \leq \|G_\tau\| \to 0 \). Also by Lemma 2.1.6 and Proposition 2.3.3 \( U_y Z_\tau, R_\tau U_y Z_\tau \) and \( V_\tau U_y Z_\tau \) tend weakly to zero, and as \( Z_\tau^{-1} U_\chi, Z_\tau^{-1} U_\chi R_\tau \) and \( Z_\tau^{-1} U_\chi V_\tau \) are uniformly bounded, the result comes by lemmas 2.1.5 and 2.1.8.

**Proposition 3.4.6.** The cosets \( \Phi_\infty^J(W^0(g)) \) with \( g \in C(\hat{\mathbb{R}}) \) are in the center of \( \mathcal{A}_\infty^J \).

**Proof:** We have that for any \( c \in PC(\hat{\mathbb{R}}) \), \( W^0(0)cI - cW^0(0) \) is compact, by Proposition [2.2.2]. We will prove now that the commutator \( W^0(0)P_\tau - P_\tau W^0(0) \) is in \( \mathcal{J} \). Write

\[
W^0(0)P_\tau - P_\tau W^0(0) = Q_\tau W^0(0)P_\tau - P_\tau W^0(0)Q_\tau.
\]

As \( Q_\tau \to 0 \) and the operators \( \chi_\pm W^0(0)\chi_\pm \) are compact we have that \( Q_\tau \chi_\pm W^0(0)\chi_\pm \) and \( \chi_\pm W^0(0)\chi_\pm Q_\tau \) converge to zero uniformly, and so the corresponding sequences belong to \( \mathcal{G} \). We are only left with the sequences \( (Q_\tau \chi_\pm W^0(0)\chi_\pm P_\tau) \) and \( (P_\tau \chi_\pm W^0(0)\chi_\pm Q_\tau) \). For them we can write

\[
Q_\tau \chi_\pm W^0(0)\chi_\pm P_\tau = V_\tau V_\tau \chi_\pm W^0(0)\chi_\pm R_\tau R_\tau
\]

and by the use of Proposition 3.3.9 these last operators are equal to \( V_\tau \chi_\pm W^0(0)J_\chi_\pm P_\tau \) with \( \chi_\pm W^0(0)J_\chi_\pm \) compact and so the sequences are in \( \mathcal{J}_1 \). Equally the sequences \( (P_\tau \chi_\pm W^0(0)\chi_\pm Q_\tau) \) by the use of Proposition 3.3.8 can be seen to be in \( \mathcal{J}_1 \), which finishes the proof.

This last result means we can localize \( \mathcal{A}_\infty^J \) over the maximal ideal space of the subalgebra \( \mathcal{C}' \) generated by the cosets \( \Phi_\infty^J(W^0(g)) \) with \( g \in C(\hat{\mathbb{R}}) \). This maximal ideal space is formed by the cosets \( \Phi_\infty^J(W^0(g_x)) \) with \( g_x(x) = 0 \) and \( x \in \hat{\mathbb{R}} \), and is isomorphic to \( \hat{\mathbb{R}} \) (that these maximal ideals are indeed not trivial, can be verified by the homomorphisms \( S_y \), defined above).

In order to apply the Local Principle of Allan, let \( \mathcal{I}_{\infty,y}, y \in \hat{\mathbb{R}} \), be the smallest closed two sided ideal of \( \mathcal{A}_\infty^J \) which contains the ideal \( y \) of \( \mathcal{C}' \). We call \( \Phi_{\infty,y} \) the homomorphism which is the composition of the canonical homomorphism from \( \mathcal{A}_\infty^J \) to \( \mathcal{A}_{\infty,y}^J := \mathcal{A}_\infty^J / \mathcal{I}_{\infty,y} \), with \( \Phi_\infty^J \). The following lemma corresponds to Lemma 3.4.2.

**Lemma 3.4.7.** If \( a \in PC(\hat{\mathbb{R}}) \) such that \( a \) is continuous at \( y \) and \( a(y) = 0 \), then \( \Phi_{\infty,y}(W^0(a)) = 0 \)
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**Proof:** For \( y \neq \infty \) let \( f_\epsilon \in C(\mathbb{R}) \) such that \( 0 \leq f < 1 \) except at the point \( y \), where \( f(y) = 1 \), and the support of \( f \) is contained in the interval \([y - \epsilon, y + \epsilon]\). We have that \( \Phi^J_{\infty,y}(W^0(f_\epsilon)) \) is the identity and then
\[
\|\Phi^J_{\infty,y}(W^0(a))\| = \|\Phi^J_{\infty,y}(W^0(a))\Phi^J_{\infty,y}(W^0(f_\epsilon))\|
\]
\[
= \|\Phi^J_{\infty,y}(W^0(a))\| \leq \|af_\epsilon\|_{L^\infty}
\]
and this last norm can be as small as desired by choosing \( \epsilon \) small enough. For \( y = \infty \) the proof is the same, but the support of \( f_\epsilon \) is contained in \( \{y \in \mathbb{R} : |y| > 1/\epsilon\} \).

3.4.3 The local algebras \( A^J_{\infty,y}, y \in \mathbb{R} \)

By Lemma 3.4.5 the strong limit \( S_y \) is an algebra homomorphism between the algebras \( A \) and \( B := \text{alg}(I, P_1, \chi_+, W^0(\chi_+)) \).

Define now the following application,
\[
S'_y : B \to A^J_{\infty,y}, \quad S'_y(A) = \Phi_{\infty,y}(U_y Z_\tau A Z^{-1}_\tau U^{-1}_y).
\]
(3.29)
The application \( S'_y \) is also an algebra homomorphism (note that \((U_y Z_\tau A Z^{-1}_\tau U^{-1}_y)_{\tau \in \mathbb{R}^+} \in A \) for any \( A \in B \) as one can see by the generators of \( B \)), and so the following result holds:

**Proposition 3.4.8.** The local algebra \( A^J_{\infty,y} \) is isomorphic to the algebra \( B \) and this isomorphism is given by \( S_y \).

**Proof:** We must first see that \( S_y \) is well defined in \( A^J_{\infty,y} \). Having already the results from Lemma 3.4.5, we have only to prove that \( S_y(A_\tau) = 0 \) for the sequences \( (A_\tau) \) in the ideal \( I_{\infty,y} \). This follows also from Lemma 3.4.5 by assertion (c), where we can see that \( S_y(W^0(a)) = 0 \) for \( a \in C(\mathbb{R}) \) such that \( a(y) = 0 \). And so it makes sense to define the quotient homomorphism
\[
S_y : A^J_{\infty,y} \to B
\]
which for simplicity is represented by the same symbol.

We are left with proving that the homomorphism \( S_y \) is really an isomorphism. If \( \Phi^J_{\infty,y}(A_\tau) \) is invertible it is easy to see that \( S_y(\Phi^J_{\infty,y}(A_\tau)) \) is also invertible, because \( S_y \) is a unital homomorphism. For the reverse direction we have that \( S'_y \) is also a unital homomorphism, and if we can prove that
\[
S'_y(S_y(\Phi^J_{\infty,y}(A_\tau))) = \Phi^J_{\infty,y}(A_\tau),
\]
(3.30)
our proof is finished. As \( S'_y, S_y \) and \( \Phi^J_{\infty,y} \) are all homomorphisms, it is sufficient to prove (3.30) for the cosets generating the algebra. But for these cosets, the result is a direct consequence of the homomorphisms definition and Lemmas 3.4.2 and 3.4.7. ■
3.4.4 The local algebra $\mathcal{A}_{\infty,\infty}^J$

By the use of Lemmas 3.4.2 and 3.4.7 it is easy to see that this algebra is generated by three projections, $\Phi_{\infty,\infty}(\chi_i)$, $\Phi_{\infty,\infty}^J(W^0(\chi_i))$ and $\Phi_{\infty,\infty}^J(P_r)$. This algebra still has a non-trivial center, as we can see in the following proposition:

**Proposition 3.4.9.** The projection $\Phi_{\infty,\infty}^J(\chi_i)$ belongs to the center of $\mathcal{A}_{\infty,\infty}^J$.

**Proof:** As $P_r$ commutes with $\chi_i$, the only relation that needs to be proved, is $\Phi_{\infty,\infty}^J(W^0(\chi_i)) = \Phi_{\infty,\infty}^J(\chi_i W^0(\chi_i))$. For this consider the function $\chi'_i$, continuous, and taking the value 0 at $-\infty$ and 1 at $+\infty$. By Lemmas 3.4.2 and 3.4.7, $\Phi_{\infty,\infty}^J(\chi_i) = \Phi_{\infty,\infty}^J(\chi'_i)$ and $\Phi_{\infty,\infty}^J(W^0(\chi_i)) = \Phi_{\infty,\infty}^J(W^0(\chi'_i))$. This together with the fact that $W^0(\chi'_i) = \chi'_i W^0(\chi'_i) + K$ for some compact operator $K$ (see Proposition 2.2.2 b) (iii)) ends the proof.

We can define the transformations $S_{\infty}^\pm r$ that to an operator $A_r$ associate the operator

$$S_{\infty}^\pm r(A_r) = \tilde{V}_{r\mp} A_r \tilde{V}_{r\pm},$$

and the strong limit

$$S_{\infty}^\pm(A_r) = s\text{-lim}_{r\to\infty} S_{\infty}^\pm r(A_r)$$

if it exists.

**Lemma 3.4.10.** The strong limits $S_{\infty}^\pm(A_r)$ exist for all $(A_r) \in \mathcal{A}$. In particular,

(a) $S_{\infty}^\pm(P_r) = \chi_{\pm}$, $S_{\infty}^\pm(Q_r) = \chi_{\pm}$;

(b) $S_{\infty}^\pm(cI) = c(\pm\infty)I$;

(c) $S_{\infty}^\pm(W^0(a)) = W^0(a)$;

(d) if $(j_r) \in \mathcal{A} \cap J$, then $S_{\infty}^\pm(j_r) \in K$.

**Proof:** The assertion (a) is proved by a simple calculation. To prove assertion (b) note that $(S_{\infty}^\pm(cI)u)(x) = c(x \pm r)u(x)$, and assertion (c) follows from $S_{\infty}^\pm(W^0(a)) = W^0(a)$. To prove the last assertion note again that for $G_r \in G$, $\|S_{\infty}^\pm(G_r)\| \leq \|G_r\| \to 0$. For $(j_r) \in J_0$, note that $\tilde{V}_{r}$ and $\tilde{V}_{r}$ tend weakly to zero as $r \to \infty$. In relation to $(j_r) \in J_1$, we have the strong limits $R_r \tilde{V}_{r} \to \chi_{\pm}^J$, $V_r \tilde{V}_{r} \to \chi_{\pm}$ and the weak limits $\tilde{V}_{r} R_r \to J \chi_{\pm}$, $\tilde{V}_{r} V_r \to \chi_{\pm}$ when $r \to +\infty$. By Lemma 2.1.6 this means that $S_{\infty}^\pm(j_r)$ tends weakly to a compact operator $K \in K$. But as $(j_r)$ also belongs to $\mathcal{A}$ and for this algebra by (a), (b) and (c) the strong limit is well defined, we conclude that $S_{\infty}^\pm(j_r)$ tends strongly to $K$.

Localizing over the maximal ideal space of the central subalgebra generated by the identity and $\Phi_{\infty,\infty}^J(\chi_i)$, let $\mathcal{I}_{\infty,\infty}^\pm$ denote the smallest closed two sided ideal in $\mathcal{A}_{\infty,\infty}^J$ containing the maximal ideal (of the central subalgebra) $\mathcal{I} \Phi_{\infty,\infty}^J(\chi_i)$, and put $\mathcal{A}_{\infty,\infty}^J = \mathcal{A}_{\infty,\infty}^J \cap \mathcal{I}_{\infty,\infty}^\pm$. A suitable closure of $\mathcal{A}_{\infty,\infty}^J$ is $\mathcal{A}_{\infty,\infty}^J$. We define the transformation $S_{\infty}^\pm$ for operators $A_{\infty}^J$ in $\mathcal{A}_{\infty,\infty}^J$ by $S_{\infty}^\pm(A_{\infty}^J) = S_{\infty}^\pm r(A_r)$ for all $(A_r) \in \mathcal{A}$.
$A^J_{\infty,\infty}/\mathcal{I}_{\infty,\infty\pm}$ with $\Phi^J_{\infty,\infty\pm}$ the canonical homomorphism from $A$ into $A^J_{\infty,\infty\pm}$. These two local algebras are generated each by two idempotents\footnote{For example, $A^J_{\infty,\infty+}$ is generated by $\Phi^J_{\infty,\infty+}(P_r)$ and $\Phi^J_{\infty,\infty+}(W^0(\chi_+))$.} and so we can apply the Two Projections Theorem (see, for instance, [17, Theorem 1.10]). This theorem gives us symbol mappings $N^\pm$, to a space of matrix functions,

$$
N^\pm(\Phi^J_{\infty,\infty\pm}(\chi_\pm)) = e^j : x \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};
$$

$$
N^\pm(\Phi^J_{\infty,\infty\pm}(P_\tau)) = p^j_1 : x \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};
$$

$$
N^\pm(\Phi^J_{\infty,\infty\pm}(W^0(\chi_+))) = p^j_2 : x \mapsto \begin{bmatrix} x \\ \sqrt{x(1-x)} \\ 1-x \end{bmatrix};
$$

with

$$
x \in \sigma_{A^J_{\infty,\infty\pm}}(\Phi^J_{\infty,\infty\pm}(P_rW^0(\chi_+)), P_\tau),
$$

if only $\{0, 1\} \subset \sigma_{A^J_{\infty,\infty\pm}}(\Phi^J_{\infty,\infty\pm}(P_rW^0(\chi_+))+P_\tau)$ and these are not isolated points of the spectrum. So we are left with finding these local spectra.

To find the local spectra, we will use the strong limits $S^\pm_{\infty}$, which will be seen to be well defined homomorphisms acting on the local algebras $A^J_{\infty,\infty\pm}$. We start by remarking that, by Lemma 3.4.10, $S^\pm_{\infty}(A \cap J) \subset K$ and $S^\pm_{\infty}(I_{\infty}) = 0$. Then $S^\pm_{\infty}$ are well defined from $A^J_{\infty}$ into the Calkin algebra $\mathcal{N}^\sigma$, with $\mathcal{N}$ generated by the convolution operators $W^0(a)$ with $a \in PC(\mathbb{R})$ and the operator of multiplication by $\chi_+$. If $f \in C(\mathbb{R})$ then $W^0(f)$ is in the center of this algebra, and we can localize via the Local Principle of Allan. The maximal ideal space of this central subalgebra is isomorphic to $\mathbb{R}$, and to a point $x$ of $\mathbb{R}$ it corresponds the maximal ideal of all operators $W^0(f)$ with $f(x) = 0$. Let $I_x$ denote the smallest closed two sided ideal in $\mathcal{N}^\sigma$ containing the ideal $x$. The set $S^\pm_{\infty}(I_{\infty,\infty})$ is then included in $I_{\infty}$ and so the homomorphisms $S^\pm_{\infty}$ are well defined from the local algebra $A^J_{\infty,\infty}$ into $\mathcal{N}^\sigma : = \mathcal{N}/I_{\infty}$. Finally, as $S^\pm_{\infty}(I_{\infty,\infty\pm}) = 0$, $S^\pm_{\infty}$ is well defined in the algebra $A^J_{\infty,\infty\pm}$ respectively.

The above result means that for any element $A_\tau$ in $A$,

$$
\sigma_{A^J_{\infty,\infty\pm}}(\Phi^J_{\infty,\infty\pm}(A_\tau)) \supset \sigma_{\mathcal{N}^\sigma_{\infty}}(S^\pm_{\infty}(A_\tau)).
$$

Applying this to our particular case we obtain, noting that $S^\pm_{\infty}(P_rW^0(\chi_+)) = \chi_+W^0(\chi_+)\chi_+$,

$$
[0, 1] \subset \sigma_{A^J_{\infty,\infty\pm}}(\Phi^J_{\infty,\infty\pm}(P_rW^0(\chi_+)+P_\tau)).
$$

To prove that the inclusion is really an equality, is only necessary to remark that

$$
\|\Phi^J_{\infty,\infty\pm}(P_rW^0(\chi_+))\| \leq \|P_rW^0(\chi_+)\| \leq 1
$$

(which means the points of the spectrum must have module less or equal to 1) and that the element $\Phi^J_{\infty,\infty\pm}(P_rW^0(\chi_+)+P_\tau)$ is the image of the positive sequence $(P_rW^0(\chi_+)+P_\tau)$ (which means the spectrum is contained in the positive half-axis).
So define the homomorphisms,

$$M^\pm = N^\pm \Phi_{\infty,\infty \pm}.$$  \hspace{1cm} (3.38)

We just proved the following result:

**Proposition 3.4.11.** The local algebras $A^J_{\infty,\infty \pm}$ are isomorphic to the unital $C^*$-subalgebra of the algebra of 2 by 2 matrix functions defined on $[0,1]$, which are diagonal at $\{0,1\}$ (and which is generated by $e'$, $p'_1$ and $p'_2$), and these isomorphisms are given by $N^\pm$. The coset $\Phi^J_{\infty,\infty \pm}(A_\tau)$ is invertible in the local algebra $A^J_{\infty,\infty \pm}$ if and only if $M^\pm(A_\tau)$ are invertible. In particular

$$M^\pm(cI) : x \mapsto \begin{bmatrix} c(\pm \infty) & 0 \\ 0 & c(\pm \infty) \end{bmatrix};$$

$$M^\pm(P_\tau) : x \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};$$

$$M^\pm(W^0(a)) : x \mapsto a(-\infty) \left[ \begin{array}{cc} 1 - x & -\sqrt{x(1-x)} \\ -\sqrt{x(1-x)} & x \end{array} \right] + a(+\infty) \left[ \begin{array}{cc} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{array} \right].$$

So we have solved the problem of identification of the local algebras for the algebra $A$, and now we can turn our attention to the larger algebra that contains the flip, the algebra $A'$.  

### 3.5 The algebra with a flip, multiplication and convolution operators

To the generators of $A$ is possible to add the flip $J$, due to the result below, which is immediate and implies that the flip belongs to our essentialization algebra $F$.

**Proposition 3.5.1.** The following relations hold:

$$R_\tau JR_\tau \to J \quad R_\tau JV_\tau = 0 \quad V_{-\tau}JR_\tau = 0 \quad V_{-\tau}JV_\tau = J$$

So let $A'$ be the $C^*$-subalgebra of $F$ generated by the constant sequences $(cI)$, $(W^0(a))$, $(J)$ with $a, c \in PC(\mathbb{R})$ and the sequence $(P_\tau)$. Designate by $A'^J$ the quotient algebra

$$\frac{A'}{A' \cap J} \cong \frac{A' + J}{J}.$$  \hspace{1cm} (3.39)

Note that $A^J \subset A'^J$. 

3.5.1 First localization

This first part is very similar to the case of the algebra without the flip. Let \( \tilde{C}(\mathbb{R}) \) represent the subspace of \( C(\mathbb{R}) \) constituted by the even functions.

**Proposition 3.5.2.** The cosets \((fI) + J\) with \( f \in \tilde{C}(\mathbb{R})\) belong to the center of \( A^J \).

**Proof:** As we have that \( P_\tau fI = fP_\tau \), \( cfI = fcI \) and \( fJ = JfI \), the only thing that is left are the convolution operators \( W^0(a) \). As Proposition 2.2.2 shows that \( fW^0(a) - W^0(a)fI \) is a compact operator, the result follows.

So we can also apply here the Local Principle of Allan, and localize \( A^J \) over the central subalgebra \( C \) generated by the set of cosets \( \{(fI) + J, f \in \tilde{C}(\mathbb{R})\} \). The maximal ideal space of this subalgebra is isomorphic to \([0, \infty]\), with a maximal ideal consisting of the cosets \( \{(fxI) + J, : fx \in \tilde{C}(\mathbb{R}), f(\pm x) = 0\} \). Let \( I'_x \) denote the smallest closed two-sided ideal in \( A^J \) containing the ideal \( x \) of \( C \). Define the local algebras \( A^J_x := A^J/I'_x \) and let \( \Phi^J_x \) (instead of \( \Phi^J_x \), for simplicity) denote the (canonical) homomorphism from \( A^J \) to \( A^J_x \).

We can now obtain Lemma 3.4.2 (with the modification that the function must be continuous and take the value 0 at the points \( \{-x, x\} \)), Proposition 3.4.3 and Corollary 3.4.4 in a very similar way to the case without the flip, so we will not repeat the arguments. These three results give the invertibility conditions we need for all the local algebras except the local algebra at infinity. But for the sake of completeness, we proceed here with the identification. For this purpose we will use the homomorphism

\[
O_x(A_\tau) = \lim_{\tau \to \infty} Z_\tau V_{-x}A_\tau V_x Z_\tau^{-1}, \quad (3.40)
\]

which is well defined in the algebra \( A^J \) (without the flip), by the following proposition.

**Lemma 3.5.3.** The strong limits \( O_x(A_\tau) \) exist for all \((A_\tau) \in A \) and \( x \in \mathbb{R} \). In particular,

(a) \( O_x(P_\tau) = I \);

(b) \( O_x(cI) = c(x^-)\chi_- + c(x^+)\chi_+ \) for \( c \in PC(\tilde{R}) \);

(c) \( O_x(W^0(a)) = a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+) \) for \( a \in PC(\tilde{R}) \);

(d) if \((j_\tau) \in J_0 \) or \((j_\tau) \in J_1 \), then \( O_x(j_\tau) = 0 \).

**Proof:** Assertion (a) is proved by simple calculation. For (b) write

\[
(Z_\tau V_{-x}cV_x Z_\tau^{-1}u)(t) = (cV_x Z_\tau^{-1}u)(t/n + x_n) = c(t/n + x)V_x Z_\tau^{-1}u(t/n + x) = c(t/n + x)u(t).
\]
Now this multiplication operator \( c(t/n + x) \) tends to \( c(x^-)\chi_+ + c(x^+)\chi_- \) (it is possible to use the same proof as in Proposition 2.3.2). Assertion (c) comes by noting that \( \hat{V}_xW^0(a)V_x = W^0(a) \) (by Proposition 2.3.2 (d), (e)) and then applying assertions (c) and (a) of the same proposition, taking into account that the Fourier transform is a bounded operator. Regarding (d), for elements of \( \mathcal{G} \) the result comes due to the uniform boundness of the operators involved. Also, as \( Z_\tau V_{-\tau}, Z_\tau V_xR_\tau \) and \( Z_\tau V_{-\tau}V_\tau \) are uniformly bounded and \( \hat{V}_xZ_\tau^{-1}, R_\tau\hat{V}_xZ_\tau^{-1} \) and \( V_{-\tau}\hat{V}_xZ_\tau^{-1} \) tend weakly to zero by Lemma 2.1.6 and Proposition 2.3.3 the result follows due to lemmas 2.1.8 and 2.1.5.

In the case \( x = 0 \), \( O_x \) is even defined in \( \mathcal{A}_{0J}^0 \), which, together with the next Proposition, will permit us to identify \( \mathcal{A}_{0J}^0 \).

**Proposition 3.5.4.** For any \( x \in \mathbb{R}^+_0 \) and \( a \in PC(\mathbb{R}) \), we have

\[
\Phi_x^J(W^0(a)) = \Phi_x^J(a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+)).
\]

**Proof:** Let \( f \in \tilde{C}(\mathbb{R}) \) such that \( f(\pm x) = 1 \) and \( f(\pm \infty) = 0 \). Then \( \Phi_x^J(fI) \) is the identity in the local algebra and we can write (put \( a' = a(-\infty)\chi_- + a(+\infty)\chi_+ \)),

\[
\Phi_x^J(W^0(a)) = \Phi_x^J(fW^0(a)) = \Phi_x^J(fW^0(a')) + \Phi_x^J(fW^0(a - a')).
\]

As \( f(\pm \infty) = (a - a')(\pm \infty) = 0 \), \( fW^0(a - a') \) is compact (see Proposition 2.2.2) and the result follows.

### 3.5.2 The local algebra \( \mathcal{A}_{0J}^0 \)

**Proposition 3.5.5.** The homomorphism \( O_0 \) is well defined in \( \mathcal{A}_{0J}^0 \). Moreover the local algebra \( \mathcal{A}_{0J}^0 \) is isomorphic to the closed algebra \( \text{alg}(I, \chi_+, W^0(\chi_+), J) \), and this isomorphism is given by \( O_0 \). In particular \( O_0(W^0(a)) = a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+) \), \( O_0(cI) = c(0^-)\chi_- + c(0^+)\chi_+ \), \( O_0(J) = J \) and \( O_0(P_\tau) = I \).

**Proof:** Regarding the particular values of \( O_0 \) for the generators of \( \mathcal{A}_{0J}^0 \), it is only necessary to remember the proposition correspondent to 3.4.3 (for \( \Phi_{0J}^0(P_\tau) \)) and remark that \( (Z_\tau cZ_\tau^{-1})(x) = c(x/\tau), Z_\tau W^0(a)Z_\tau^{-1} = W^0(a_\tau) \), with \( a_\tau(\xi) = a(\xi\tau) \) and \( Z_\tau JZ_\tau^{-1} = J \). This gives immediately that \( O_0(I_0^J) = \{0\} \) and so \( O_0 \) is well defined in \( \mathcal{A}_{0J}^0 \). Now the proof that \( O_0 \) is an isomorphism is very similar to the proof of Proposition 3.4.8 with the inverse being defined as \( O_0^J(A) = \Phi_{0J}^0(A) \).

### 3.5.3 The local algebras \( \mathcal{A}_{xJ}^0, \ x \in \mathbb{R}^+ \)

To identify these local algebras we will eliminate the flip by doubling the dimension (see Lemma 2.4.1).
3.5. THE ALGEBRA WITH A FLIP, MULTIPLICATION AND...

So let \( f_x \) be a continuous function with support in \( \mathbb{R}^+ \) such that \( f_x(x) = 1 \), and put \( p = \Phi_x^\tau(f_x), j = \Phi_x^\tau(J) \) and \( e = \Phi_x^\tau(I) \). We have that \( p^2 = p, p \) commutes with all the algebra generators except \( j \) and \( j p j = e - p \). Any element \( a \in \mathcal{A}_x^\tau \) can be written (due to the properties of \( J \)) as \( a = a_1 + a_2 j \), where \( a_i \) belongs to the algebra without the flip. It is then possible to apply Lemma 2.4.1

Define now the homomorphism

\[
\hat{O}_x: \mathcal{A}_x^\tau \to [\text{alg}(I, \chi_+, W^0(\chi_+))]^{2 \times 2}, \quad \hat{O}_x = O_x L
\]

(3.41)

where \( O_x \) represents now the canonical (diagonal) extension for matrix operators of the strong limit defined in (3.40). We have then

**Proposition 3.5.6.** The local algebra \( \mathcal{A}_x^\tau, \ x \in \mathbb{R}^+ \) is isomorphic to \([\text{alg}(I, \chi_+, W^0(\chi_+))]^{2 \times 2}\) and this isomorphism is given by \( \hat{O}_x \). In particular

\[
\hat{O}_x (\Phi_x^\tau(p)) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},
\]

(3.42)

\[
\hat{O}_x (\Phi_x^\tau(W^0(a))) = \begin{bmatrix} a(-\infty)W^0(\chi_+) + a(+\infty)W^0(\chi_+) & 0 \\ 0 & a(+\infty)W^0(\chi_+) + a(-\infty)W^0(\chi_+) \end{bmatrix},
\]

(3.43)

\[
\hat{O}_x (\Phi_x^\tau(cI)) = \begin{bmatrix} c(x^-) \chi_+ + c(x^+) \chi_+ & 0 \\ 0 & c(-x^+) \chi_+ + c(-x^-) \chi_+ \end{bmatrix},
\]

(3.44)

\[
\hat{O}_x (\Phi_x^\tau(J)) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.
\]

(3.45)

**Proof:** To prove the specific values of \( \hat{O}_x \), use Lemma 2.4.1 together with the Proposition corresponding to Proposition 3.4.3 (for \( \Phi_x^\tau(p) \)), and note that \( Z_\tau \tilde{V}_x W^0(a) \tilde{V}_x Z_\tau^{-1} = W^0(a_\tau) \) with \( a_\tau(\xi) = a(\xi \tau) \), and that \( (Z_\tau \tilde{V}_x c \tilde{V}_x Z_\tau^{-1})(x) = c(x/\tau + x) \). These values imply that \( \hat{O}_x(I) = \{0\} \) and so \( \hat{O}_x \) is well defined in \( \mathcal{A}_x^\tau \). We have only to prove now that invertibility in \( \text{Im}(\hat{O}_x) \) is equivalent to invertibility in \( \mathcal{A}_x^\tau \). As \( \hat{O}_x \) is a unital homomorphism, if \( \Phi_x^\tau(A_\tau) \) is invertible in \( \mathcal{A}_x^\tau \), it easy to see that \( \hat{O}_x (\Phi_x^\tau(A_\tau)) \) is invertible, and due to the inverse closedness, the inverse must belong to \([\text{alg}(I, \chi_+, W^0(\chi_+))]^{2 \times 2}\). To prove the converse define the application

\[
\hat{O}_x': [\text{alg}(I, \chi_+, W^0(\chi_+))]^{2 \times 2} \to \mathcal{A}_x^\tau,
\]

\[
\hat{O}_x' \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) = L^{-1} \left( \begin{bmatrix} p\Phi_x^\tau(\tilde{V}_x A_{11} \tilde{V}_x) & p\Phi_x^\tau(\tilde{V}_x A_{12} \tilde{V}_x) \\ p\Phi_x^\tau(\tilde{V}_x A_{21} \tilde{V}_x) & p\Phi_x^\tau(\tilde{V}_x A_{22} \tilde{V}_x) \end{bmatrix} \right),
\]

(3.46)

This application is a unital homomorphism, and in a similar way to the proof of Proposition 3.4.3 it can be shown that

\[
\hat{O}_x' (\hat{O}_x (\Phi_x^\tau(A_\tau))) = \Phi_x^\tau(A_\tau),
\]

and so invertibility in \( \text{Im}(\hat{O}_x) \) implies invertibility in \( \mathcal{A}_x^\tau \).
3.5.4 Second localization

**Proposition 3.5.7.** The cosets \( \Phi_{\infty}^{J}(W^0(g)) \) with \( g \in \tilde{\mathcal{C}}(\mathbb{R}) \) are in the center of \( \mathcal{A}_\infty^{J} \).

**Proof:** It is almost immediate that \( W^0(g)J = JW^0(g) \). The other results have the same proof as those in Proposition 3.4.6. \( \blacksquare \)

This last result means that we can again localize \( \mathcal{A}_\infty^{J} \) over the maximal ideal space of the subalgebra \( \mathcal{C}' \) generated by these cosets. This maximal ideal space is now formed by the cosets \( \Phi_{\infty}^{J}(W^0(f_x)) \) with \( f_x(x) = f_x(-x) = 0 \) and \( x \in [0, \infty] \), and is isomorphic to \([0, \infty] \).

In order to apply the Local Principle of Allan, let \( \mathcal{T}_{\infty, y} \), \( y \in [0, \infty] \), be the smallest closed two sided ideal of \( \mathcal{A}_\infty^{J} \) which contains the ideal \( y \) of \( \mathcal{C}' \). We call \( \Phi_{\infty}^{J, y} \) the homomorphism which is the composition of the canonical homomorphism from \( \mathcal{A}_\infty^{J} \) to \( \mathcal{A}_\infty^{J}/\mathcal{T}_{\infty, y} \), with \( \Phi_{\infty}^{J} \). We have again a Lemma that identifies some elements of \( \mathcal{T}_{\infty, y} \) and whose proof, due to being almost the same as the proof of Lemma 3.4.7, we omit.

**Lemma 3.5.8.** If \( a \in PC(\mathbb{R}) \) such that \( a \) is continuous at \( \pm y \) and \( a(\pm y) = 0 \), then \( \Phi_{\infty, y}^{J}(W^0(a)) = 0 \)

In the next subsection, in order to identify the local algebras, we will use again the strong limits \( S_y \), defined in (3.28), but the only algebra in which we can apply the homomorphism directly is \( \mathcal{A}_\infty^{J, 0} \), because only \( S_0 \) is well defined (i.e. the strong limit exists) when applied to the flip \( J \).

3.5.5 The local algebra \( \mathcal{A}_\infty^{J, 0} \)

If we define the algebra \( \mathcal{B}' \) as \( \text{alg}(I, P_1, \chi^+, W^0(\chi^+), J) \), the strong limit \( S_0 \) is an algebra homomorphism between the algebras \( \mathcal{A}' \) and \( \mathcal{B}' \) and the following theorem, with a proof equal to the case without the flip (Prop. 3.4.8), assures that \( S_0 \) is also an isomorphism.

**Proposition 3.5.9.** The local algebra \( \mathcal{A}_\infty^{J, 0} \) is isomorphic to the algebra \( \mathcal{B}' \) and this isomorphism is given by \( S_0 \).

3.5.6 The local algebras \( \mathcal{A}_\infty^{J, y}, y \in \mathbb{R}^+ \)

We will here again apply Lemma 2.4.1 to eliminate the flip by doubling the dimension. So let \( f_y \) be a continuous function with support in \( \mathbb{R}^+ \) such that \( f_y(y) = 1 \), and put \( p = \Phi_{\infty, y}^{J}(W^0(f_y)), j = \Phi_{\infty, y}^{J}(J) \) and \( e = \Phi_{\infty, y}^{J}(I) \). We have that \( p^2 = p, p \) commutes
with all the algebra generators except $j$ and $jpj = e - p$. Any element $a \in \mathcal{A}_{\infty,y}^{T}$ can be written (due to the properties of $J$) as $a = a_1 + a_2 j$, where $a_i$ belongs to the algebra without the flip. It is then possible to apply Lemma 2.4.1. Define the following homomorphism,

$$ \hat{S}_y : \mathcal{A}_{\infty,y}^{T} \rightarrow [\text{alg}(I, \chi_+, W^0(\chi_+), P_1)]^{2 \times 2}, \quad \hat{S}_y = S_y L $$

(3.47)

where $S_y$ represents the canonical extension for matrix operators of the strong limit defined in (3.28). We have then

**Proposition 3.5.10.** The algebra $\mathcal{A}_{\infty,y}^{T}, \ y \in \mathbb{R}^+$ is isomorphic to $[\text{alg}(I, P_1, \chi_+, W^0(\chi_+))]^{2 \times 2}$ and this isomorphism is given by $\hat{S}_y$. In particular

$$ \hat{S}_y (\Phi_{\infty,y}^{T}(P_1)) = \begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix} \quad (3.48) $$

$$ \hat{S}_y (\Phi_{\infty,y}^{T}(W^0(a))) = \begin{bmatrix} a(y^-)W^0(\chi_-) + a(y^+)W^0(\chi_+) & 0 \\ 0 & a(-y^-)W^0(\chi_-) + a(-y^+)W^0(\chi_+) \end{bmatrix} \quad (3.49) $$

$$ \hat{S}_y (\Phi_{\infty,y}^{T}(cI)) = \begin{bmatrix} c(-\infty)\chi_- + c(+\infty)\chi_+ & 0 \\ 0 & c(+\infty)\chi_- + c(-\infty)\chi_+ \end{bmatrix} \quad (3.50) $$

$$ \hat{S}_y (\Phi_{\infty,y}^{T}(J)) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad (3.51) $$

**Proof:** First we must see that $\hat{S}_y$ is well defined in $\mathcal{A}_{\infty,y}^{T}$. Having already the results from Lemma 3.4.5 one can easily see that $\hat{S}_y (I_\infty)$ and $\hat{S}_y (I_{\infty,y})$ are $\{0\}$. The computation of formulas (3.48) to (3.51) is easy by the use of the results in Lemma 3.4.5. We shall only prove now, that invertibility in $\text{Im}(\hat{S}_y)$ is equivalent to invertibility in $\mathcal{A}_{\infty,y}^{T}$. As $\hat{S}_y$ is a unital homomorphism, if $\Phi_{\infty,y}^{T}(A_\tau)$ is invertible in $\mathcal{A}_{\infty,y}^{T}$, then $\hat{S}_y (\Phi_{\infty,y}^{T}(A_\tau))$ is invertible. To prove the converse define the application

$$ \hat{S}_y' : [\text{alg}(I, P_1, \chi_+, W^0(\chi_+))]^{2 \times 2} \rightarrow \mathcal{A}_{\infty,y}^{T}, $$

$$ \hat{S}_y' \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) = \quad (3.52) $$

$$ L^{-1} \left( \begin{bmatrix} p\Phi_{\infty,y}^{T}(U_yZ_\tau A_{11}Z_\tau^{-1}U_{-y}) & p\Phi_{\infty,y}^{T}(U_yZ_\tau A_{12}Z_\tau^{-1}U_{-y}) \\ p\Phi_{\infty,y}^{T}(U_yZ_\tau A_{21}Z_\tau^{-1}U_{-y}) & p\Phi_{\infty,y}^{T}(U_yZ_\tau A_{22}Z_\tau^{-1}U_{-y}) \end{bmatrix} \right). $$

This application is a unital homomorphism, and in a similar way to the proof of Proposition 3.4.8 it can be shown that

$$ \hat{S}_y' (\hat{S}_y (\Phi_{\infty,y}^{T}(A_\tau))) = \Phi_{\infty,y}^{T}(A_\tau), $$

and so invertibility in $\text{Im}(\hat{S}_y)$ implies invertibility in $\mathcal{A}_{\infty,y}^{T}$. \[\square\]
3.5.7 The local algebra $A_{J,\infty}^{\infty}$

For this local algebra we have no locally equivalent representation given by strong limits, and so the algebra must be studied by the properties of its generators.

**Proposition 3.5.11.** The local algebra $A_{J,\infty}^{\infty}$ is generated by the elements $e := \Phi_{\infty,\infty}(I)$, $p_1 := \Phi_{\infty,\infty}(P_{\tau})$, $p_2 := \Phi_{\infty,\infty}(W^0(\chi_+))$, $p_3 := \Phi_{\infty,\infty}(\chi_+)$ and $j := \Phi_{\infty,\infty}(J)$.

**Proof:** We have that

$$\Phi_{\infty,\infty}(cI) = \Phi_{\infty,\infty}(c(-\infty)\chi_- + c(+\infty)\chi_+) - \Phi_{\infty,\infty}((c - c(-\infty)\chi_- - c(+\infty)\chi_+)I)$$

and as $c - c(-\infty)\chi_- - c(+\infty)\chi_+$ is a function continuous at infinity that takes the value 0 there, by Lemma 3.4.2,

$$\Phi_{\infty,\infty}((c - c(-\infty)\chi_- - c(+\infty)\chi_+)I) = 0.$$ 

Using the same reasoning,

$$\Phi_{\infty,\infty}(W^0(a)) = \Phi_{\infty,\infty}(a(-\infty)W^0(\chi_-) + c(+\infty)W^0(\chi_+)).$$

For the other generators, the result is obvious.

We have then an algebra generated by the identity and four idempotents. The non trivial relations between these generators are given in the next proposition. The only relation that does not come directly from the already known relations in the algebra $A$ is $p_2p_3 = p_3p_2$, or $\Phi_{\infty,\infty}(W^0(\chi_+)) = \Phi_{\infty,\infty}(\chi_+ W^0(\chi_+))$. The proof in this case is the same as in Proposition 3.4.9.

**Proposition 3.5.12.** The following relations hold:

- $jp_1j = p_1$, $p_1p_3 = p_3p_1$;
- $jp_2j = e - p_2$, $p_2p_3 = p_3p_2$;
- $jp_3j = e - p_3$.

As the projection $p_3$ commutes with $p_1$ and $p_2$, and $jp_3j = e - p_3$, we are in the position again to apply Lemma 2.4.1 and eliminate the flip by doubling the dimension. This means we have an isomorphism $L : A_{J,\infty}^{\infty} \rightarrow D^{2 \times 2}$, with $D := \text{alg}(p_3, p_1p_3, p_2p_3)$, whose image for the generators of $A_{J,\infty}^{\infty}$ are

$$L(e) = \begin{bmatrix} p_3 & 0 \\ 0 & p_3 \end{bmatrix}, \quad L(p_1) = \begin{bmatrix} p_1p_3 & 0 \\ 0 & p_1p_3 \end{bmatrix}, \quad L(p_2) = \begin{bmatrix} p_2p_3 & 0 \\ 0 & (e - p_2)p_3 \end{bmatrix},$$
3.5. THE ALGEBRA WITH A FLIP, MULTIPLICATION AND...

\[ L(p_3) = \begin{bmatrix} p_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad L(j) = \begin{bmatrix} 0 & p_3 \\ p_3 & 0 \end{bmatrix}, \]  

and we can see, because \( p_3 \) is the identity for \( \mathcal{D} \), that we are in the presence of an algebra generated by the identity and two idempotents. As before, the Two Projections Theorem gives us a symbol mapping \( N \), to a space of matrix functions,

\[ N(p_3) : x \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \]  
\[ N(p_1p_3) : x \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \]  
\[ N(p_2p_3) : x \mapsto \begin{bmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{bmatrix}; \]

with \( x \in \sigma_{\mathcal{D}}(p_1p_3p_2p_3p_1) \), if only \( \{0, 1\} \subset \sigma_{\mathcal{D}}(p_1p_3p_2p_3p_1) = \sigma_{\mathcal{D}}(\Phi_{\infty,\infty}(P_T\chi_+W^0(\chi_+\chi_+P_r))). \)

So we only need to find this local spectrum.

**Proposition 3.5.13.** \( \sigma_{\mathcal{D}}(\Phi_{\infty,\infty}(P_T\chi_+W^0(\chi_+\chi_+P_r))) = [0, 1]. \)

**Proof:** By the same arguments as in the case of the algebra without the flip, one can see that

\[ \sigma_{\mathcal{D}}(\Phi_{\infty,\infty}(P_T\chi_+W^0(\chi_+\chi_+P_r))) \subset [0, 1]. \]  

To prove the other inclusion suppose that \( \lambda \in ]0, 1[ \) and \( \Phi_{\infty,\infty}(\lambda \chi_+ - P_T\chi_+, W^0(\chi_+\chi_+)\chi_+P_r) \) is invertible in \( \mathcal{D} \). Then it is not difficult to see that \( \Phi_{\infty,\infty}(\lambda I - P_T\chi_+W^0(\chi_+)\chi_+P_r) \) is invertible in \( \mathcal{A}_{\infty,\infty}^\mathcal{J} \). Now define the function \( \chi'_+ \), continuous on \( \mathbb{R} \), with \( \chi'_+(-\infty) = 0 \) and \( \chi'_+(+\infty) = 1 \), and such that the imaginary part of \( \chi'_+(x) \) is greater than zero for \( x \in \mathbb{R} \). We have that, for \( A_r = \lambda I - P_T\chi'_+W^0(\chi'_+)\chi'_+P_r \),

\[ \Phi_{\infty,\infty}(\lambda I - P_T\chi_+W^0(\chi_+)\chi_+P_r) = \Phi_{\infty,\infty}(A_r), \]  

and \( \hat{S}_s(A_r) \) (\( s \in \mathbb{R}^+ \)), \( S_0(A_r) \) are invertible. As also \( O^0((A_r)) \) is Fredholm (see [11] or [32] Section 15) we have that \( A_r + \mathcal{J} \) is invertible in \( \mathcal{A}_{\infty}^\mathcal{J} \). But this is a contradiction, since by Proposition 3.4.11 \( A_r + \mathcal{J} \) is not invertible in \( \mathcal{A}_{\infty}^\mathcal{J} \) and \( \mathcal{A}_{\infty}^\mathcal{J} \) is a *-subalgebra of \( \mathcal{A}_{\infty}^\mathcal{J} \). So we proved that the open interval \( ]0, 1[ \) must be contained in the spectrum, and as the spectrum must be closed the result follows.

To synthesize our results, let \( M' = NL \) and \( M = M'\Phi_{\infty,\infty}^\mathcal{J} \). Let \( \mathcal{Z} \) be the closed subalgebra of the 4 by 4 matrix functions defined in the interval \( [0, 1] \) generated by the
elements:

\[
e' : x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad (3.59)
\]

\[
p'_1 : x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad (3.60)
\]

\[
p'_2 : x \mapsto \begin{bmatrix} x & \sqrt{x(1-x)} & 0 & 0 \\ \sqrt{x(1-x)} & 1 - x & 0 & 0 \\ 0 & 0 & 1 - x & -\sqrt{x(1-x)} \\ 0 & 0 & -\sqrt{x(1-x)} & x \end{bmatrix}; \quad (3.61)
\]

\[
p'_3 : x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad (3.62)
\]

\[
j' : x \mapsto \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (3.63)
\]

We proved the following proposition.

**Proposition 3.5.14.** The local algebra \( A'_{\infty,\infty} \) is isometrically isomorphic to the algebra \( Z \) and this isomorphism is given by \( M' \). The coset \( \Phi_{\infty,\infty}(A_\tau) \) is invertible in the local algebra \( A'_{\infty,\infty} \) if and only if \( M'(A_\tau) \) is invertible in \( Z \). In particular we have:

- \( M(cI) = c(-\infty)(e' - p'_3) + c(+\infty)p'_3; \)
- \( M(W^0(a)) = a(-\infty)(e' - p'_3) + a(+\infty)p'_2; \)
- \( M(J) = j'; \)
- \( M(P_\tau) = p'_1. \)

### 3.5.8 The main theorem

Having identified all the local algebras, we can now state the main result. The detailed proof of this result is written above (is the subject of this whole section), and so below we just give a sketch of the proof.
Theorem 3.5.15. Let $A$ be any operator from the subalgebra of $\mathcal{L}(L^2(\mathbb{R}))$ generated by operators $cI$, $W^0(a)$, with $a, c \in PC(\mathbb{R})$ and the flip operator $J$, with $(Ju)(x) = u(-x)$. Then the approximation method

$$(P_\tau AP_\tau + Q_\tau)u_\tau = f$$

can be used to get an approximate solution to the equation

$$Au = f$$

(i.e. the method applies to $A$) if and only if $A, \tilde{A}, S_0(A_\tau), M(A_\tau)$ and $\tilde{S}_s(A_\tau), s \in \mathbb{R}^+$ are invertible, with $A_\tau = P_\tau AP_\tau + Q_\tau$.

Proof: By Theorems 3.2.1 and 3.3.3 the applicability of the approximation method is equivalent to the invertibility of the operators $A$ and $\tilde{A}$, and the invertibility of the coset $(A_\tau) + J$ in $\mathcal{A}(\mathcal{J})$. The cosets $(fI) + J$ with $f \in \tilde{C}(\mathbb{R})$ are in the center of $\mathcal{A}(\mathcal{J})$ and applying the Local Principle of Allan, instead of an invertibility problem in $\mathcal{A}(\mathcal{J})$, we have now invertibility problems in each of the (simpler) local algebras $\mathcal{A}_x^\mathcal{J}$, $x \in [0, +\infty[$. All the local algebras except the local algebra at infinity are identifiable and if $A$ is a Fredholm operator then the cosets $\Phi_t^\mathcal{J}(A_\tau), t \in \mathbb{R}_0^+$ are invertible (see Propositions 3.5.5 and 3.5.6). Regarding the local algebra $\mathcal{A}_\infty^\mathcal{J}$, it has the cosets $\Phi_{g}^\mathcal{J}(W^0(g)), g \in C(\mathbb{R})$ in its center. Applying again the Local Principle of Allan, we are able to identify all the local algebras through the isomorphisms $S_0$ (the local algebra at the point 0 - see Proposition 3.5.9), $M$ (the local algebra at the point infinity - see Proposition 3.5.14), and $\tilde{S}_y$ (the local algebras corresponding to the points $y \in \mathbb{R}^+$ - see Proposition 3.5.10), obtaining in this way the aimed result. 

3.6 The system case

Formally, we only proved Theorem 3.5.15 for the scalar case. But if we consider $a$ or $c$ in $[PC(\mathbb{R})]^{n \times n}$, the proofs remain the same. This covers the operators related with systems of singular integral equations or systems of Wiener-Hopf operators. Obviously the operators that result from the homomorphisms have then matrix coefficients, and can be difficult in the general case to find invertibility conditions for these operators. For a non scalar version of the Two Projections Theorem see [14].

3.7 Examples

We will continue our exposition on the Finite Section Method by presenting two examples of application of Theorem 3.5.15. The first is about Singular Integral Operators,
where the results concerning the Finite Section Method are already known (see, for example, [17]), and the other is the application to an operator of Wiener-Hopf plus Hankel type, which, to the best of the authors’s knowledge, appeared the first time due to the results above. In what follows the symbols $P_1$ and $Q_1$ can also represent the matrix operators with $P_1$ and $Q_1$ in the main diagonal and zero elsewhere, when needed.

### 3.7.1 Singular integral operators

Consider the operator

$$A = c_1 W^0(\chi_+) + c_2 W^0(\chi_-)$$

(3.64)

with $c_{1,2} \in PC(\mathbb{R})$. This operator is usually called a paired singular integral operator, as it can be written in the form

$$\frac{c_1 + c_2}{2} I + \frac{c_1 - c_2}{2} S_{\mathbb{R}}$$

Applying Theorem 3.5.15 to $A$ we obtain the following result:

**Corollary 3.7.1.** The approximation method (3.1) applies to the operator $A$ if and only if the operator $A$ is invertible and the point 0 is not contained in the area limited by the triangle defined by the points $1, \frac{c_1(-\infty)}{c_2(-\infty)}$ and $\frac{c_1(+\infty)}{c_2(+\infty)}$.

**Proof:** The direct use of Theorem 3.5.15 leads to the following operators (resp. operator functions) that must be invertible, in order to guarantee the applicability of the approximation method:

(a) $A = c_1 W^0(\chi_+) + c_2 W^0(\chi_-)$;

(b) $\tilde{A} = (c_1(-\infty)\chi_- + c_1(+\infty)\chi_+) W^0(\chi_-) + (c_2(-\infty)\chi_- + c_2(+\infty)\chi_+) W^0(\chi_+)$;

(c) $S_0(A_\tau) = Q_1 + P_1((c_1(-\infty)\chi_- + c_1(+\infty)\chi_+) W^0(\chi_-) + (c_2(-\infty)\chi_- + c_2(+\infty)\chi_+) W^0(\chi_+)) P_1$;

(d) $S_y(A_\tau) = Q_1 + P_1 \begin{pmatrix} (c_1(-\infty)\chi_- + c_1(+\infty)\chi_+) & 0 \\ 0 & (c_2(+\infty)\chi_- + c_2(-\infty)\chi_+) \end{pmatrix} P_1$;

(e) $M(A_\tau)$:

$$x \mapsto \begin{bmatrix} c_2(+\infty)(1-x) + c_1(+\infty)x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_2(-\infty)x + c_1(-\infty)(1-x) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
for \( x \in [0, 1] \).

The fact that \( A \) must be invertible implies immediately the invertibility of \( \tilde{A} \) and that \( c_{1,2}(\pm \infty) \) are different from zero, which also means that \( \tilde{S}_y(A_\tau) \) are invertible. It is also not difficult to see that the invertibility of \( S_0(A_\tau) \) implies the invertibility of \( M(A_\tau) \) Finally writing \( S_0(A_\tau) \) as the operator

\[
d'W^0(x_+) + W^0(x_-), \quad d'(x) = \begin{cases} \frac{c_1(+\infty)}{c_2(+\infty)} & 0 < x < 1 \\ \frac{c_1(-\infty)}{c_2(-\infty)} & -1 < x < 0 \\ 1 & |x| > 1 \end{cases}
\]

and applying the usual invertibility conditions for singular integral operators (see [22]) it is obtained that the point 0 must not be contained in the area limited by the triangle defined by the points \( 1, \frac{c_1(-\infty)}{c_2(-\infty)} \) and \( \frac{c_1(+\infty)}{c_2(+\infty)} \).

We can consider also a more complex type of paired operator. Let \( A \) be the operator

\[
A = W^0(a)x_+ + W^0(b)x_-
\]

with \( a, b \in PC(\mathbb{R}) \). The following result is then obtained:

**Corollary 3.7.2.** The approximation method 3.1 applies to the operator \( A \) in (3.65) if and only if the following conditions are satisfied:

(a) The operator \( A \) is invertible;

(b) The operator \( x_+W^0(\tilde{a})x_+ + x_-W^0(\tilde{b})x_- \) is invertible;

(c) For any \( y \in \mathbb{R} \), the point 0 is not contained in the area limited by the triangle defined by the points \( \{1, \frac{a(y^-)}{a(y^+)}; \frac{b(y^-)}{b(y^+)}\} \).

**Proof:** As the operator \( A \) belongs to the algebra without the flip we can apply the more simple homomorphisms \( S_y \) instead of \( \tilde{S}_y \) (as could have been done in the previous example). A necessary and sufficient condition for the applicability of the finite section method is the invertibility of the following operators (resp. operator functions):

(i) \( A = W^0(a)x_+ + W^0(b)x_- \);

(ii) \( \tilde{A} = x_+W^0(\tilde{a})x_+ + x_-W^0(\tilde{b})x_- \);

(iii) \( S_y(A_\tau) = Q_1 + P_1 \left( W^0(x_-)(a(y^-)x_+ + b(y^-)x_-) + W^0(x_+)(a(y^+)x_+ + b(y^+)x_-) \right) P_1 \)

for \( y \in \mathbb{R} \);
(iv) \(M(A_r)\) :

\[
x \mapsto \begin{bmatrix}
a(-\infty)(1-x) + a(+\infty)x & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & b(-\infty)x + b(+\infty)(1-x) & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix},
\]

for \(x \in [0, 1]\).

The first and second conditions of the Corollary come directly from (i) and (ii). The third condition is seen to be equivalent to (iii), by writing \(S_y(A_r)\) as the operator

\[
W^0(\chi_-)a' + W^0(\chi_+), \quad a'(x) = \begin{cases} 
a(y^-) & 0 < x < 1 \\
a(y^+) & -1 < x < 0 \\
1 & |x| > 1 \end{cases},
\]

and applying the usual invertibility conditions for singular integral operators (see [22]). Finally, the invertibility of \(A\) implies the invertibility of \(M(A_r)\). 

### 3.7.2 Wiener-Hopf plus Hankel operators

Now let \(A\) be the operator

\[
A = \chi_+ W^0(a)\chi_+ + \chi_+ W^0(b)J\chi_+ + \chi_-,
\]

with \(a, b \in PC(\mathbb{R})\). Before giving the result, we must first introduce some notation.

For any pair of complex numbers \(\xi, \eta\) define the set \(\Omega(\xi, \eta)\) in the following way. Let \(r_\xi\) and \(r_\eta\) denote the half-lines starting at the origin and passing through the points \(-i\xi\) and \(-i\eta\) respectively. If these two half-lines coincide, then \(\Omega(\xi, \eta)\) is the whole complex plane except the part of \(r_\xi\) whose points have module greater or equal than \(2\sqrt{|\xi\eta|}\). If they do not coincide, then consider the branch of hyperbola with \(r_\xi\) and \(r_\eta\) as asymptotes and that passes through the point \(-i(\xi + \eta)\). This branch divides the complex plane into two components, and \(\Omega(\xi, \eta)\) is the open component that contains the origin. The applicability of the Finite Section Method for operator \(A\) is expressed in the following result.

**Corollary 3.7.3.** The approximation method \([3.1]\) applies to \(A\) if and only if the following conditions hold:

(a) The operator \(A\) is invertible;
(b) the operator \(\chi_+ W^0(\bar{a})\chi_+ + \chi_-\) is invertible;
(c) \(b(0^-) - b(0^+) \in \Omega(a(0^-), a(0^+))\);
(d) for any \( y \in \mathbb{R}^+ \) the roots of the polynomial

\[
\xi^2 - \left( \frac{a(y^+)}{a(y^-)} + \frac{a(-y^+)}{a(-y^-)} + \frac{b(y^+)}{a(y^-)} - \frac{b(y^-)}{a(y^-)} \right) \xi + \frac{a(y^+)}{a(y^-)}a(-y^+) \]

satisfy the inequality

\[
|\arg \frac{a(y^+)}{a(y^-)} + \arg \frac{a(-y^+)}{a(-y^-)} - \arg \xi| < \pi
\]

**Proof:** The direct use of Theorem 3.5.15 leads to the following operators (resp. operator functions) that must be invertible, in order for the approximation method to apply:

(i) \( A = \chi_+ W^0(a)\chi_+ + \chi_+ W^0(b)J\chi_+ + \chi_-; \)

(ii) \( \tilde{A} = \chi_+ W^0(\tilde{a})\chi_+ + \chi_-; \)

(iii) \( S_0(A_\tau) = P_1 (\chi_+ (a(0^-)W^0(\chi_-) + a(0^+)W^0(\chi_+))\chi_+ + \chi_+ (b(0^-)W^0(\chi_-) + b(0^+)W^0(\chi_+))J\chi_+ + \chi_-) P_1 + Q_1; \)

(iv) \( S_y (A_\tau) = P_1 \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} P_1 + Q_1, \) with

\[
\begin{align*}
S_{11} &= \chi_+ (a(y^-)W^0(\chi_-) + a(y^+)W^0(\chi_+))\chi_+ + \chi_- \\
S_{12} &= \chi_+ (b(y^-)W^0(\chi_-) + b(y^+)W^0(\chi_+))\chi_+ \\
S_{21} &= \chi_- (b(-y^-)W^0(\chi_-) + b(-y^+)W^0(\chi_+))\chi_+ \\
S_{22} &= \chi_- (a(-y^-)W^0(\chi_-) + a(-y^+)W^0(\chi_+))\chi_+ + \chi_-;
\end{align*}
\]

(v) \( M(A_\tau) : x \mapsto \begin{bmatrix} a(-\infty)(1-x) + a(+\infty)x & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \) \( x \in [0,1]. \)

So conditions (a) and (b) come directly from (i) and (ii), and (v) is also easily seen to be satisfied if \( \tilde{A} \) is invertible (condition (ii) again). Regarding (iii) and (iv), in the general setting, conditions for the invertibility of these type of operators that are both exact and easy to verify are still not known. But in the scalar case, which we are dealing here with, there can be applied the conditions obtained in [42, Theorem 4], which give directly the result enunciated in the corollary.
Chapter 4

Spline approximation methods with uniform meshes

4.1 Introduction

In this chapter we will try to obtain conditions for the applicability of approximation methods using polynomial splines and uniform meshes. The fact that we restrict ourselves here to uniform meshes will permit us to apply the powerful machinery based on the so-called De Boor’s estimates and developed in [17]. Roch [30], using and improving this machinery, treated the infinite Galerkin method with smoothest splines for Wiener-Hopf operators with piecewise continuous symbols. Here we will study finite (or infinite) Galerkin or collocation methods with maximum defect splines applied to algebras generated by convolution and multiplication operators both with piecewise continuous symbols. It was possible to avoid here some of the problems regarding unidentified compact operators appearing (see [17], [30]) which can only be solved by “cutoff techniques”, by the use of maximum defect splines. The author was also able to treat finite meshes by introducing the “cutoff” at infinity in the form of a finite section projection with the number \( \tau_n \) giving the “cutoff” point.

For \( n \in \mathbb{N} \) consider the mesh sequence \( (\Delta^n) \), with

\[
\Delta^n = \{ x^n_j = j/n, -n\tau_n \leq j \leq n\tau_n \}
\]

where the \( \tau_n \) are given positive numbers and define

\[
J^n_j = \left\lfloor \frac{j}{n}, \frac{j + 1}{n} \right\rfloor.
\]

Our objective is to study projections associated with this mesh and the piecewise polynomials space

\[
S^n = \{ u \in L^2(\mathbb{R}) : u_{\mid_{J^n_j}} \in P^d(J^n_j), u_{\mid_{-\infty, -n\tau_n]}(\cdot)\mid_{n\tau_n, +\infty} = 0 \}
\]
where \( \mathbb{P}^d(Y) \) represent the set of polynomials of degree less or equal to \( d \) defined on \( Y \subset \mathbb{R} \). For technical reasons we will consider a decomposition of these projections into the projection associated with the infinite mesh sequence

\[
\Delta^n = \{ x^{(n)}_j = j/n, -\infty \leq j \leq +\infty \}
\]

with the corresponding spline space \( S^n \) and the finite section projection \( P_\tau \), with \( \tau = \tau_n \) a function of \( n \) such that \( \tau_n \to +\infty \) when \( n \to +\infty \) and \( n\tau_n \) is (a positive) integer. Note that now our sequences are indexed on the positive integers and not on the continuous set of last chapter. But all the essential results remain unchanged, because what matters are the limits when the index is large, and the behaviour is maintained with integer indexes.

In this chapter we will consider a special type of basis for the spline space. Let \( \{ u^l_{jn}, 0 \leq l \leq d, -\infty < j < +\infty \} \) be an orthonomal basis for \( S^n \) such that

\[
u^l_{jn}(x) = \sqrt{n} u^l(nx - j),
\]

with \( \text{supp}(u^l) \subset [0, 1] \), \( u^l \) real valued and \( u^0 = \chi_{[0,1]} \). Except for the normalization factor \( \sqrt{n} \), this is a basis of the type of the ones studied in [17]. The functions \( u^l \) form what will be called the “mother splines”. Of all the projections into \( S^n \) that will be studied, the Galerkin projection defined by

\[
(P^n u)(x) = \sum_{j=-\infty}^{+\infty} \sum_{l=0}^{d} \int_{\Delta^n_j} u(y) u^l_{jn}(y) \, dy \ u^l_{jn}(x) \quad (4.1)
\]

will play a fundamental role, because it is bounded in \( L^2(\mathbb{R}) \) and is the selfadjoint projection onto \( S^n \). The orthogonal projection \( I - P^n \) will be denoted by \( Q^n \).

An important concept in approximation methods, as we already saw in the last chapter is the concept of stability of a sequence. To recall the definition, we say that a sequence \( (A_n)_{n \in \mathbb{N}} \) with \( A_n : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) uniformly bounded is stable if \( A_n \) is invertible for all \( n \) sufficiently large and the norms \( \| A_n^{-1} \| \) are uniformly bounded as \( n \) goes to infinity. As in Theorem 3.2.1 it is possible to prove that if an operator \( A \) is invertible, and there exists a stable sequence \( (A_n) \) that tends strongly to \( A \), then for \( n \) large the solution \( u_n \) of the equation

\[
A_n u_n = v, \quad v \in L^2(\mathbb{R}) \quad (4.2)
\]

exists, and \( u_n \) tends (in the \( L^2 \) norm) to the solution \( u \) of the equation

\[
A u = v. \quad (4.3)
\]

For a projection \( L^n \) that is not bounded in the whole of \( L^2(\mathbb{R}) \) (like a collocation projection) it is still valid that if \( (A_n) \) is stable then the solution \( u_n \) of

\[
A_n u_n = L^n v, \quad \text{where} \ A_n = L^n A_n, \quad (4.4)
\]

approximates the function \( u \), provided that \( L^n v \) and \( A_n|_{L^n} \) make sense (are bounded) and tend to \( v \) and \( A \) respectively. We say then that the approximation method is applicable to \( A \) (see [17, Section 1.3.1]).
In order to study stability, we will as before reduce the problem to one of invertibility in a given $C^*$-algebra and this time make use of some interesting and important relationships between the sequences and block Laurent operators acting on $l^2_{d+1}$ (see the introduction for the definitions).

Following [17], we will associate with each $(d + 1)$-$l^2$-vector a function in $S^n$ and vice-versa. Let $E_n : l^2_{d+1} \rightarrow S^n$ be defined by

$$E_n(X_0, \ldots, X_d) = \sum_{l=0}^{d} \sum_{j=-\infty}^{+\infty} x_{jl}u^l_{jn}$$

(4.5)

and $E_{-n} : S^n \rightarrow l^2_{d+1}$ as

$$E_{-n}(\sum_{l=0}^{d} \sum_{j=-\infty}^{+\infty} x_{jl}u^l_{jn}) = (X_0, \ldots, X_d).$$

(4.6)

Both these operators have norms independent of $n$ and equal to 1, and are in the literature usually referred as De Boor’s Estimates (see [17]). We have now these four important results:

**Theorem 4.1.1 (Commutator).** Let $f$ be a uniformly continuous function on $\mathbb{R}$ and $a$ a continuous matrix-valued function acting on the unit circle. Then

$$\|E_n T^0(a) E_{-n} P^n f I - f E_n T^0(a) E_{-n} P^n\|_{\mathcal{L}(L^2)} \rightarrow 0$$

as $n \rightarrow \infty$.

**Proof:** In [17] Section 2.12.4] this result is proved for $f \in C(\mathbb{R})$, but the proof only makes use of the continuity of $f$ in $\mathbb{R}$, and the fact that the module of continuity of $f$, defined as $\omega(f, \delta) = \sup_{|x-y|<\delta} |f(x) - f(y)|$ tends to zero as $\delta \rightarrow 0$. Thus the proof there is valid for functions $f$ that are uniformly continuous. □

**Theorem 4.1.2.** The operator $E_{-n} P^n S_{\mathbb{R}} E_n : l^2_{d+1} \rightarrow l^2_{d+1}$ is independent of $n$ and equal to $T^0(\sigma_P)$, with $\sigma_P$ a piecewise continuous function on $\mathbb{T}$ which is continuous at all points except the point 1, and verifies at this point the equation ($\sigma^0_P$ represents the $li$ entry of the matrix function $\sigma_P$)

$$\sigma^0_P(1+) - \sigma^0_P(1-) = -2,$$

$$\sigma^0_P(1+) - \sigma^0_P(1-) = 0, \quad 0 \leq l, i \leq d, \quad l \neq 0 \text{ or } i \neq 0.$$

**Proof:** See Theorem 2.15 in [17]. □

For some convolution operators it can be shown also the following result, which was proved for $d = 0$ in [30] (see also section 2.11.2 in [17]).
Theorem 4.1.3. Let \( g \) be a continuous function on \( \mathbb{R} \) with compact support contained in \([-m, m]\), \( m > 0 \). Then, given \( d \geq 0 \) and a mother spline basis \( \{u^l, 0 \leq l \leq d\} \) there exists a continuous function \( g_n^0 \in [C(\mathbb{T})]^{d+1,d+1} \) given by

\[
g_n^{li}(e^{2\pi i y}) = \begin{cases} 
  g(-ny)F\tilde{u}^i(y)Fu^l(y) & \text{if } 0 \leq y \leq \frac{m}{n} \\
  0 & \text{if } \frac{m}{n} < y < 1 - \frac{m}{n} \quad (4.7) \\
  g(n(1-y))F\tilde{u}^i(y-1)Fu^l(y-1) & \text{if } 1 - \frac{m}{n} \leq y < 1,
\end{cases}
\]

for \( n > 2m \) and \( \tilde{u} \) defined as \( \tilde{u}(x) = u(-x) \), such that \( E_{-n}P^0W^0(g)E_n = T^0(g_n^i) \).

**Proof:** The \( kj \)-entry of the matrix corresponding to the operator

\[
(E_{-n}P^0W^0(g)E_n)^{li} : l_2 \to l_2, \quad 0 \leq i, l \leq d
\]
is represented by \( \sigma_{kjn}^{li} \). This coefficient can be written as

\[
\sigma_{kjn}^{li} = \int_{-\infty}^{+\infty} (W^0(g)u^i_{jn}(x))u^l_{kn}(x) \, dx = \int_{-\infty}^{+\infty} \left( \frac{1}{\sqrt{n}} W^0(g)u^i_{jn}(\frac{x+k}{n}) \right) u^l(x) \, dx
\]

and because

\[
\frac{1}{\sqrt{n}} (W^0(g)u^i_{jn})(\frac{x+k}{n}) = W^0(g_n)u^i(x+k-j)
\]

where \( g_n(\xi) = g(n\xi) \) depends on the difference \( k-j \), it is possible to conclude that we are in the presence of a (matrix) Laurent operator, with the Fourier coefficients of the generator function given by

\[
\sigma_{kjn}^{li} = \int_{-\infty}^{+\infty} (W^0(g_n)u^i)(x+k)u^l(x) \, dx.
\]

Then it is possible to write

\[
\sigma_{kn}^{li} = \int_{-\infty}^{+\infty} (W^0(g_n)u^i)(x+k)u^l(x) \, dx = \int_{-\infty}^{+\infty} (W^0(\tilde{g}_n)\tilde{u}^i)(-k-x)u^l(x) \, dx
\]

\[
= (W^0(\tilde{g}_n)\tilde{u}^i) \ast u^l(-k) = F^{-1} ((F(W^0(\tilde{g}_n)\tilde{u}^i))(Fu^l))(-k) =
\]

\[
= F^{-1} (\tilde{g}_nF\tilde{u}^iFu^l)(-k) = \int_{-\infty}^{+\infty} e^{-2\pi i k \xi} g(-n\xi)F\tilde{u}^i(\xi)Fu^l(\xi) \, d\xi \quad (4.8)
\]

(note that no convergence problems occur since the support of \( g \) is compact) and by introducing new variables \( s \in \mathbb{Z}, \ y \in [0,1] \) with \( \xi = s + y \) we obtain

\[
\int_{-\infty}^{+\infty} e^{-2\pi i k \xi} g(-n\xi)F\tilde{u}^i(\xi)Fu^l(\xi) \, d\xi =
\]

\[
= \sum_s \int_0^1 e^{-2\pi i y k} g(-n(y+s))F\tilde{u}^i(y+s)Fu^l(y+s) \, dy =
\]

\[
= \int_0^1 e^{-2\pi i y k} \left( \sum_s g(-n(y+s))F\tilde{u}^i(y+s)Fu^l(y+s) \right) \, dy \quad (4.9)
\]
4.2. ALGEBRAIZATION

and so getting the function \( e^{2\pi iy} \to \sum_s g(-n(y+s))F\hat{u}^i(y+s) Fu^i(y+s) \) as the generating function of the Laurent operator. Moreover, if \( n > 2m \), all the terms of the series are zero except for \( s = -1 \) and \( s = 0 \), and one obtains (4.7), and \( g_n^\dagger \) is continuous because the Fourier transforms and \( g \) are continuous.

**Theorem 4.1.4.** If \( a \in [PC(\mathbb{T})]^{d+1,d+1} \) then the strong limit of \( E_n T^0(a) E_{-n} P^n \) exists and is equal to

\[
\frac{a_{00}(1+) + a_{00}(1-)}{2} I - \frac{a_{00}(1+) - a_{00}(1-)}{2} S_{\mathbb{R}}.
\]

**Proof:** See Proposition 3.13 in [17], taking into consideration that by the choice of our spline basis we have (in that book’s notation) \( \alpha^F = (1,0,\ldots,0)^T \). ■

4.2 Algebraization

Let now \( \mathcal{E} \) be the set formed by all the sequences of operators \( (A_n)_{n \in \mathbb{N}}, \ A_n : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \), such that \( \sup_n \| A_n \|_{L(L^2)} < \infty \). This set with the operations \( (A_n) + (B_n) = (A_n + B_n) \) and \( (A_n)/(B_n) = (A_n B_n) \), with the norm \( \| (A_n) \| = \sup_n \| A_n \|_{L(L^2)} \) and involution \( (A_n)^* = (A_n^*) \) is a unital \( C^* \)-algebra.

Let \( \mathcal{G} \) be the closed two-sided ideal in \( \mathcal{E} \) of all sequences \( (A_n) \) for which we have \( \lim_{n \to \infty} \| A_n \| = 0 \). Write \( \Phi \) for the canonical homomorphism \( \mathcal{E} \to \mathcal{E}/\mathcal{G} \) and \( \| \Phi \| \) for the norm in the quotient algebra.

The following proposition is known (see for instance Theorem 3.2.1 or [17, Proposition 1.2]) and will reduce the stability problem to an invertibility problem:

**Proposition 4.2.1.** Let \( (A_n) \in \mathcal{E} \). The sequence \( (A_n) \) is stable iff the coset \( \Phi(A_n) \) is invertible in \( \mathcal{E}/\mathcal{G} \).

4.3 Essentialization

As before, we will restrict now our algebra. Let \( \mathcal{F} \subset \mathcal{E} \) be the set of all sequences \( (A_n) \) for which there exist operators \( A, A_{ij}, i,j = 1,2 \) such that the following strong limits as \( n \to \infty \) exist:

- \( A_n \to A \) and \( A_n^* \to A^* \);
- \( R_{\tau_n} A_n R_{\tau_n} \to A_{11} \) and \( (R_{\tau_n} A_n R_{\tau_n})^* \to A_{11}^* \);
- \( R_{\tau_n} A_n V_{\tau_n} \to A_{12} \) and \( (R_{\tau_n} A_n V_{\tau_n})^* \to A_{12}^* \);
Using the fact that $I = R_{\tau_n} R_{\tau_n} + V_{\tau_n} V_{-\tau_n}$ and Lemma 2.3.1, one can see that this set is actually a closed $C^*$-subalgebra of $\mathcal{E}$ which contains $G$.

Now let $K \subset L^2(\mathbb{R})$ denote the ideal of compact operators and define $J_0$ and $J_1$ to be the sets

\[ J_0 = \{(K) + (G_n), \ K \in \mathcal{K}, \ (G_n) \in G\}, \]

\[ J_1 = \{(R_{\tau_n} K_1 R_{-\tau_n} + R_{\tau_n} K_2 V_{-\tau_n} + V_{\tau_n} K_3 R_{-\tau_n} + V_{\tau_n} K_4 V_{-\tau_n}) + (G_n), \ K_k \in \mathcal{K}, \ (G_n) \in G\}. \]

The proof of the following proposition is the same as in the last chapter.

**Proposition 4.3.1.** $J_0$ and $J_1$ are closed two sided ideals of $\mathcal{F}$.

So it is possible now to define the $*-$homomorphisms in $\mathcal{F}$

\[
O^0 : \mathcal{F} \to \mathcal{L}(L^2), \ O^0((A_{\tau_n})) = A \tag{4.10}
\]

\[
O^1 : \mathcal{F} \to \mathcal{L}(L^2)^{2 \times 2}, \ O^1((A_{\tau_n})) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{4.11}
\]

It is not difficult to see that $O^i(G) = 0$ ($i \in \{0, 1\}$), that

\[
O^0((K)) = K,
\]

\[
O^0((R_{\tau_n} K_1 R_{-\tau_n} + R_{\tau_n} K_2 V_{-\tau_n} + V_{\tau_n} K_3 R_{-\tau_n} + V_{\tau_n} K_4 V_{-\tau_n})) = 0,
\]

and

\[
O^1((K)) = 0,
\]

\[
O^1((R_{\tau_n} K_1 R_{-\tau_n} + R_{\tau_n} K_2 V_{-\tau_n} + V_{\tau_n} K_3 R_{-\tau_n} + V_{\tau_n} K_4 V_{-\tau_n})) = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}
\]

for every $K, K_k \in \mathcal{K}$.

The sets in $\mathcal{F}/G$, $\{j_0 + G, \ j_0 \in J_0\}$ and $\{j_1 + G, \ j_1 \in J_1\}$ are naturally closed two sided ideals and so let $\mathcal{J}$ denote the smallest closed two-sided ideal in $\mathcal{F}$ containing both $J_0$ and $J_1$.

We are now in the conditions to apply the Lifting Theorem [17, Theorem 1.8] and obtain the result:
Theorem 4.3.2 (Lifting). The coset \((A_n) + G\) is invertible in the quotient algebra \(\mathcal{F}/G\) if and only if the operators \(O^0((A_n)), O^1((A_n))\) are invertible in \(\mathcal{L}(L^2)\) (resp. \(\mathcal{L}(L^2)^{2 \times 2}\)) and the coset \((A_n) + J\) is invertible in \(\mathcal{F}^J := \mathcal{F}/J\).

It is at this point necessary to verify that the sequences we are interested in belong to the algebra \(\mathcal{F}\). In the last chapter were found several strong limits as \(\tau \to \infty\) for the flip, convolution and multiplication operators. These results remain true, with \(\tau\) substituted by \(\tau_n\) and so it is possible to immediately conclude the following result.

Proposition 4.3.3. The constant sequences \((J), (cI), (W^0(a))\) with \(a, c \in PC(\mathbb{R})\), and the sequence \((P_{\tau_n})\) are in \(\mathcal{F}\).

We must see now if the sequences which have to do with the splines are also in \(\mathcal{F}\). For that we define the following (diagonal) operators acting on \(l^2_{d+1}\), with \(k\) a natural number and \(X = (x_j)_{j \in \mathbb{Z}}\) (as they are diagonal, for simplification purposes we write the definition in a "scalar" way):

\[
\begin{align*}
(\hat{P}_k X)_j &= \begin{cases} 
  x_j & \text{if } -k \leq j < k \\
  0 & \text{if } j \geq k \\
  0 & \text{if } j < -k
\end{cases}, \\
(\hat{R}_k X)_j &= \begin{cases} 
  0 & \text{if } 0 \leq j < k \\
  x_{k-1-j} & \text{if } -k \leq j < 0 \\
  x_{k-1-j} & \text{if } j \geq 0 \\
  0 & \text{if } j < 0 \\
  0 & \text{if } j \geq 0
\end{cases}, \\
(\hat{V}_k X)_j &= \begin{cases} 
  0 & \text{if } -k \leq j < k \\
  x_{j-k} & \text{if } j \geq k \\
  x_{j-k} & \text{if } j < -k \\
  0 & \text{if } j \geq 0 \\
  0 & \text{if } j < 0
\end{cases}, \\
(\hat{V}_{-k} X)_j &= \begin{cases} 
  0 & \text{if } -k \leq j < k \\
  0 & \text{if } j \geq k \\
  x_{j-k} & \text{if } j < -k \\
  0 & \text{if } j \geq 0 \\
  0 & \text{if } j < 0
\end{cases}.
\end{align*}
\]

The relations in Lemma 2.3.1 are still valid for the corresponding "hat" operators. And it is also not difficult to see that

\[
\begin{align*}
\hat{P}_{\tau_n} &= E_n P_{\tau_n} E_n \\
\hat{R}_{\tau_n} &= E_n R_{\tau_n} E_n \\
\hat{R}'_{\tau_n} &= E_n R'_{\tau_n} E_n \\
\hat{V}_{\tau_n} &= E_n V_{\tau_n} E_n \\
\hat{V}_{-\tau_n} &= E_n V_{-\tau_n} E_n \\
\hat{\mathcal{P}} &= E_n \chi_\omega E_n \\
\hat{J} &= E_n J E_n.
\end{align*}
\]

We are now in the conditions to prove the following result:
Proposition 4.3.4. The sequences \((E_nT^0(a)E_{-n}P^n + Q^n)\) with \(a \in [PC(\mathbb{T})]^{d+1,d+1}\) are in \(\mathcal{F}\).

Proof: The strong limit of \(E_nT^0(a)E_{-n}P^n\) exists by Theorem 4.1.4 above. And given the relations from Lemma 2.3.1 and (4.12) the proofs of the existence of the other strong limits follow closely the proofs for the corresponding strong limits for the convolution operator \(W^0(a)\) (see the last chapter, Propositions 3.3.7-3.3.10). As an example, put \(d = 0\) and consider the strong limit \(\chi_+R_{\tau_n}(E_nT^0(a)E_{-n}P^n + Q^n)R_{\tau_n}\chi_+\). As \(Q^n\) commutes with \(R_{\tau_n}\) (as do \(V_{\tau_n}\) and \(V_{-\tau_n}\)) the second term equals \(\chi_+P_{\tau_n}Q^n\) and converges strongly to zero. We are left with \(\chi_+R_{\tau_n}E_nT^0(a)E_{-n}P^nR_{\tau_n}\chi_+\). Using (4.12) it is possible to write

\[
\chi_+R_{\tau_n}E_nT^0(a)E_{-n}P^nR_{\tau_n}\chi_+ = E_n\hat{R}_{\tau_n}\hat{P}T^0(a)\hat{P}\hat{R}_{\tau_n}E_{-n}P^n,
\]

and is not difficult to see that

\[
\hat{R}_{\tau_n}\hat{P}T^0(a)\hat{P}\hat{R}_{\tau_n} = \hat{P}_{\tau_n}\hat{P}T^0(\hat{a})\hat{P}\hat{P}_{\tau_n},
\]

where \(\hat{a}(t) = a(1/t)\). Using again (4.12) we obtain finally that

\[
\chi_+R_{\tau_n}E_nT^0(a)E_{-n}P^nR_{\tau_n}\chi_+ = \chi_+P_{\tau_n}E_nT^0(\hat{a})E_{-n}P^nP_{\tau_n}\chi_+
\]

and the strong limit exists by Theorem 4.1.4. If we consider the strong limit \(\chi_+R_{\tau_n}E_nT^0(a)E_{-n}P^nR_{\tau_n}\chi_+\), using the same ideas it is possible to verify that

\[
\chi_+R_{\tau_n}E_nT^0(a)E_{-n}P^nR_{\tau_n}\chi_- = E_n\hat{R}_{\tau_n}\hat{J}QT^0(\hat{a})\hat{Q}\hat{R}_{\tau_n}E_{-n}P^n =
\]

\[
= E_n\hat{R}_{\tau_n}\hat{J}QT^0(\hat{a})\hat{Q}\hat{R}_{\tau_n}E_{-n}P^n = R_{\tau_n}'J\chi_-E_nT^0(\hat{a})E_{-n}P^n\chi_-P_{\tau_n}
\]

and this last sequence tends strongly to zero. Calculating by the same method the limits corresponding to \(\chi_-R_{\tau_n}E_nT^0(a)E_{-n}P^nR_{\tau_n}\chi_+\) and \(\chi_-R_{\tau_n}E_nT^0(a)E_{-n}P^nR_{\tau_n}\chi_-\) we finally obtain the limit for \(R_{\tau_n}E_nT^0(a)E_{-n}P^nR_{\tau_n}\). The proof of the other strong limits is similar, and as the "hat" operators are diagonal, the extension to the matrix case is possible without difficulty.

So let now the algebra \(\mathcal{A} \subset \mathcal{F}\) be generated by the finite section projection sequence \((P_{\tau_n})\), the sequences \((E_nT^0(\sigma)E_{-n}P^n + Q^n)\) with \(\sigma \in [C(\mathbb{T})]^{d+1,d+1}\) and such that \(\sigma(1)\) is a (scalar) constant times the identity\(^1\) the constant sequences \((cI)\) with \(c \in PC_Z(\mathbb{R})\) and the constant sequences \((W^0(a))\) with \(a \in PC(\mathbb{R})\).

Designate by \(\mathcal{A}/\mathcal{J}\) the quotient algebra

\[
\frac{\mathcal{A}}{\mathcal{A} \cap \mathcal{J}} \cong \frac{\mathcal{A} + \mathcal{J}}{\mathcal{J}}.
\]

(4.13)

This algebra, as the ones appearing in the last chapter, has a rich center, and we will again make use of it through localization in order to obtain invertibility criteria for the elements.

\(^1\)From now on, we will write \(\sigma \in [C(\mathbb{T})]^{d+1,d+1}\).

CHAPTER 4. UNIFORM MESHES
4.4 First localization and identification

In order to be able to localize using the cosets \((fI) + J\) with \(f \in C(\mathbb{R})\), it is necessary first to verify that these cosets are in the center of \(A^J\). The multiplication operator \(fI\) commutes trivially with \(P_{\tau_n}\) and \(cI\). The commutator \(fW^0(a) - W^0(a)fI\) is a compact operator (see Proposition 2.2.2), so the corresponding (constant) sequence is in \(J\). And by the Commutator Theorem above we have that \(E_nT^0(\sigma)E_{-n}P^n fI - fE_nT^0(\sigma)E_{-n}P^n\) tends in the norm to zero. Note that for \(\sigma = I\), \(E_nT^0(\sigma)E_{-n}P^n + Q^n\) is the identity operator and so \(Q^n = I - E_nT^0(I)E_{-n}P^n\).

We introduce now the following modified transformation \(O_{x_n}\) such that to the sequence \((A_n)\) associates

\[
O_{x_n}(A_n) = Z_n \tilde{V}_{-x_n} A_n \tilde{V}_{x_n} Z_n^{-1}
\]

where \(x_n\) is the mesh point in \(\Delta^n\) nearer to the point \(x\). We have that \(|x - x_n| < 1/n\), and for \(x\) integer we even have \(x_n = x\). Now define the homomorphism

\[
O_x(A_n) = \text{s-lim}_{n \to \infty} O_{x_n}(A_n)
\]

for the sequences for which it exists.

**Lemma 4.4.1.** The strong limits \(O_x(A_n)\) exist for all \((A_n) \in A\) and \(x \in \mathbb{R}\). In particular,

\( (a) \) \(O_x(P_{\tau_n}) = I\), \(O_x(E_nT^0(\sigma)E_{-n}P^n + Q^n) = E_1T^0(\sigma)E_{-1}P^1 + Q^1\) for \(\sigma \in [C(\mathbb{T})]^{d+1,d+1}_1\);

\( (b) \) \(O_x(cI) = c(x^-)\chi_\_ + c(x^+)\chi_+\) for \(c \in PC_2(\mathbb{R})\);

\( (c) \) \(O_x(W^0(a)) = a(-\infty)W^0(\chi_\_) + a(+\infty)W^0(\chi_+)\) for \(a \in PC(\mathbb{R})\);

\( (d) \) if \((j_n) \in J_0\) or \((j_n) \in J_1\), then \(O_x(j_n) = 0\).

**Proof:** The first part of (a) is proved by simple calculation. For the second part note that \(\tilde{V}_{-x_n} P^n \tilde{V}_{x_n} = P^n\), \(E_{-n}\tilde{V}_{x_n}E_n\) is a corresponding "hat" shift operator acting on \(l^2_{d+1}\) and so \(E_{-n} \tilde{V}_{-x_n} E_n T^0(\sigma)E_{-n} \tilde{V}_{x_n} E_n = T^0(\sigma)\). On the other hand we have \(Z_n P^n Z_n^{-1} = P^1\) and \(Z_n E_n = E_1\), \(E_{-n} Z_n^{-1} = E_{-1}\). For (b) write

\[
(Z_n \tilde{V}_{-x_n} c \tilde{V}_{x_n} Z_n^{-1} u)(t) = (c \tilde{V}_{x_n} Z_n^{-1} u)(t/n + x_n) = c(t/n + x_n) \tilde{V}_{x_n} Z_n^{-1} u(t/n + x_n) = c(t/n + x_n) u(t).
\]

Now, if \(x\) is integer, this multiplication operator \(c(t/n + x_n)\) tends to \(c(x^-)\chi_\_ + c(x^+)\chi_+\) as \(x_n = x\). If \(x\) is not integer, the function \(c\) is continuous in a neighbourhood of \(x\), and so the multiplication operator \(c(t/n + x_n)\) tends to \(c(x)I\). Assertion (c) is proved as in the Proposition 3.5.3. Regarding (d), for elements of \(G\) the result comes due to the uniform boundedness of the operators involved. Also, as in the proof of Proposition 2.3.3 it is possible to see that \(R_{\tau_n} \tilde{V}_{x_n} Z_n^{-1}\) and \(V_{-\tau_n} \tilde{V}_{x_n} Z_n^{-1}\) tend weakly to zero, and this together with lemmas 2.1.8 and 2.1.5 proves the result.
Using the homomorphism $O_x$ it is now possible to see that the commutative subalgebra of $A^J$ generated by the cosets $(fI) + J$ is isomorphic to the algebra generated by the multiplication operators $fI$ ($f$ always in $C(\mathbb{R})$) and the maximal ideal space is isomorphic to $\mathbb{R}$. We use then localization. Designate by $\mathcal{I}_x$ the smallest closed two-sided ideal of $A^J$ containing $x$, by $A^J_x$ the quotient (local) algebra $A^J/\mathcal{I}_x$, and by $\Phi^J_x$ the canonical homomorphism onto $A^J_x$. We obtain then the following identification results, the first of them with the same proof as Lemma 3.4.2.

**Lemma 4.4.2.** If $c \in PC_Z(\mathbb{R})$ such that $c$ is continuous at $x$ and $c(x) = 0$, then $\Phi^J_x(cI) = 0$.

**Lemma 4.4.3.** If $a \in PC(\mathbb{R})$ and $c \in PC_Z(\mathbb{R})$ then the identities $\Phi^J_x(cI) = \Phi^J_x(c(x^-)\chi_- + c(x^+)\chi_+)$, $\Phi^J_x(P_{\tau_n}) = \Phi^J_x(I)$, and $\Phi^J_x(W^0(a)) = \Phi^J_x(a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+))$ hold.

**Proof:** The first identity comes directly from Lemma 4.4.2 above. For $\Phi^J_x(P_{\tau_n})$, let $y$ be greater then $|x|$ and define $f_x$ to be a continuous function supported in the interval $]y, y[$ such that $f_x(x) = 1$. Then $\Phi^J_x(f_xI)$ is the identity in the local algebra we have $\Phi^J_x(Q_\tau) = \Phi^J_x(f_xI)\Phi^J_x(Q_\tau) = \Phi^J_x(f_xQ_\tau)$. As $\|f_xQ_\tau\| \to 0$, we conclude that $\Phi^J_x(Q_\tau) = 0$, which means that $\Phi^J_x(P_{\tau}) = \Phi^J_x(I-Q_\tau) = I-\Phi^J_x(Q_\tau) = I$. For the last identity note that by Proposition 2.2.2 the operator $f_x(W^0(a) - a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+))$ is compact.

**Proposition 4.4.4.** The local algebra $A^J_x$ is isomorphic to the algebra $B = \text{alg}(E_1T^0(\sigma)E_{-1}P^1, \chi_+, W^0(\chi_-))$ if $x$ is integer, and is isomorphic to the subalgebra of $B$, $\text{alg}(E_1T^0(\sigma)E_{-1}P^1, W^0(\chi_+))$ if $x$ is non integer. This isomorphism is given by $O_x$.

**Proof:** Due to Lemma 4.4.1 above, one can easily see that $O_x$ is a well defined unital homomorphism in $A^J_x$ and has image $B$ if $x$ is integer, or the subalgebra of $B$ in the other case. To prove that $O_x$ is an isomorphism it is only necessary to find its inverse. Let $O'_x$ be the homomorphism $O'_x : B \to A^J_x$, $O'_x(A) = \Phi^J_x(V_{x_n}Z_n^{-1}AZ_nV_{-x_n})$.

Note that $O'_x$ is well defined because $V_{x_n}Z_n^{-1}AZ_nV_{-x_n} \in A$ for all $A \in B$ as can be seen by its generators. To see that it is surjective use Lemma 4.4.3 above. It is then not difficult to see that $O'_x$ is indeed the inverse of $O_x$.

We are left with the local algebra at infinity.

### 4.5 Second localization and identification

**Lemma 4.5.1.** If $g \in C(\mathbb{R})$ and $g(\infty) = 0$, then the sequences $(Q^nW^0(g))$ and $(W^0(g)Q^n)$ are in $G$. 


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Proposition 4.5.2. If \( g \in \mathcal{C}(\mathbb{R}) \) then the coset \( \Phi_J^J(W^0(g)) \) commutes with all elements of \( \mathcal{A}_J^J \).

Proof: For \( c \in \mathcal{P}C(\mathbb{R}) \) the commutator \( W^0(g) cI - cW^0(g) \) is compact by Proposition 2.2.2 and so the corresponding sequence is in \( \mathcal{J} \). Regarding the Galerkin projection \( P^n \), with help of Lemma 4.5.1 by writing \( W^0(g) = g(\infty)I + W^0(g_0) \) with \( g_0(\infty) = 0 \), one can see that the sequence \( (Q^nW^0(g) - W^0(g)Q^n) \) is in \( \mathcal{G} \). In Proposition 3.4.6 was already proved that \( (P_nW^0(g) - W^0(g)P_n) \) is in \( \mathcal{J} \). Finally, for the cosets related to \( (E_nT^0(\sigma)E_{-n}P^n) \) write \( W^0(g) = g(\infty)I + W^0(g - g(\infty)) \). We have then to prove the result only for functions \( g \) such that \( g(\infty) = 0 \). These functions can be approximated by functions with compact support. But then Theorem 4.1.3 affirms that \( E_{-n}P^nW^0(g)E_n = T^0(g_n^\dagger) \), where \( g_n^\dagger \) is in \( [\mathcal{C}(\mathbb{T})]^{d+1,d+1} \). By applying this theorem we obtain

\[
(E_nT^0(\sigma)E_{-n}P^nW^0(g)) = (E_nT^0(\sigma)E_{-n}P^nW^0(g)P^n) + \mathcal{G} = (E_n\sigma(1)T^0(g_n^\dagger)E_{-n}P^n) + (E_nT^0(\sigma - \sigma(1))T^0(g_n^\dagger)E_{-n}P^n) + \mathcal{G}
\]

with \( \sigma(1) = c_1I \) for some complex constant \( c_1 \), and as \( \sigma - \sigma(1) \) is 0 at the point 1, for \( n \) large the norm of the function \( \sigma - \sigma(1) \) is small and tends to zero (see formula 4.7), which means that the corresponding sequence is in \( \mathcal{G} \). So it is possible to write

\[
(E_nT^0(\sigma)T^0(g_n^\dagger)E_{-n}P^n) + \mathcal{G} = (E_nT^0(g_n^\dagger)T^0(\sigma)E_{-n}P^n) + \mathcal{G} = (P^nW^0(g)E_nT^0(\sigma)E_{-n}P^n) + \mathcal{G} = (W^0(g)E_nT^0(\sigma)E_{-n}P^n) + \mathcal{G},
\]

and this ends the proof.
As was done for the first localization, we introduce now an algebra homomorphism. Let the application $S_{yn}$ be such that to a sequence $(A_n) \in \mathcal{A}$ associates

$$S_{yn}(A_n) = Z_{-n}^{-1}U_yA_nU_{-y}Z_{n}. \quad (4.17)$$

For the sequences for which it exists, we define the strong limit

$$S_y(A_n) = s\lim_{n \to -\infty} S_{yn}(A_n) \quad (4.18)$$

**Lemma 4.5.3.** If $(A_n) \in \mathcal{A}$, then the limit $S_y(A_n)$ exists. In particular,

(a) $S_y(P_{\tau_n}) = P_1$, $S_y(E_nT^0(\sigma)E_{-n}P^n + Q^n) = \sigma(1)I$ for $\sigma \in [C(T)]_{d+1, d+1}$;

(b) $S_y(cI) = c(-\infty)\chi_-- c(\infty)\chi_+$ for $c \in PC_Z(\mathfrak{R})$;

(c) $S_y(W^0(a)) = a(y^-)W^0(\chi_-) + a(y^+)W^0(\chi_+)$ for $a \in PC(\mathfrak{R})$;

(d) If $(j_n) \in \mathcal{J}_0$ or $(j_n) \in \mathcal{J}_1$, then $S_y(j_n) = 0$.

**Proof:** For the first part of (a) note that $U_{-y}P_{\tau_n}U_y = P_{\tau_n}$ and then $Z_{-n}^{-1}P_{\tau_n}Z_{n} = P_1$. For the second part we have that $(U_{-y}(E_nT^0(\sigma)E_{-n}P^n + Q^n)U_y) = (E_nT^0(\sigma)E_{-n}P^n + Q^n)Z_{n} = E_{n\tau_n}T^0(\sigma)E_{-n\tau_n}P^{n\tau_n} + Q^{n\tau_n}$ and it is possible to apply Theorem 4.1.4. The other assertions have the same proof as in Proposition 3.4.5.

So we can localize $\mathcal{A}_{\infty}^J$ over the maximal ideal space of the subalgebra $\mathcal{C}$ generated by the cosets $\Phi_{\infty}^J(W^0(g))$ with $g \in C(\mathfrak{R})$. This maximal ideal space is formed by the cosets $\Phi_{\infty}^J(W^0(g_x))$ with $g_x(x) = 0$ and $x \in \mathfrak{R}$, and is isomorphic to $\mathfrak{R}$, as can be seen with the help of the homomorphism $S_y$.

In order to apply the Local Principle of Allan, let $T_{\infty, y}$, $y \in \mathfrak{R}$, be the smallest closed two sided ideal of $\mathcal{A}_{\infty}^J$ which contains the ideal $y$ of $\mathcal{C}$. We call $\Phi_{\infty, y}^J$ the homomorphism which is the composition of the canonical homomorphism from $\mathcal{A}_{\infty}^J$ to $\mathcal{A}_{\infty, y}^J := \mathcal{A}_{\infty}^J/T_{\infty, y}$, with $\Phi_{\infty}^J$. We have now the following results regarding the local algebras $\mathcal{A}_{\infty, y}^J$, the first of them having the same proof as Lemma 3.4.7.

**Lemma 4.5.4.** If $a \in PC(\mathfrak{R})$ such that $a$ is continuous at $x$ and $a(x) = 0$, then $\Phi_{\infty, x}^J(W^0(a)) = 0$.

**Lemma 4.5.5.** If $y \in \mathfrak{R}$, then the local algebra $\mathcal{A}_{\infty, y}^J$ is generated by the cosets $\Phi_{\infty, y}^J(I)$, $\Phi_{\infty, y}^J(P_{\tau_n})$, $\Phi_{\infty, y}^J(\chi_+)$ and $\Phi_{\infty, y}^J(W^0(\chi_+))$.

**Proof:** Due to Lemma 4.5.4 we obtain that

$$\Phi_{\infty, y}^J(W^0(a)) = \Phi_{\infty, y}^J(a(y^-)W^0(\chi_-)) + a(y^+)W^0(\chi_+).$$
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A similar argument to that used in Lemma 4.4.3 proves that

$$\Phi^J_{\infty,y}(cI) = \Phi^J_{\infty,y}(c(-\infty)\chi_+ + c(+\infty)\chi_+)$$

for $c \in PC_{\tilde{R}}$. Now write

$$\Phi^J_{\infty,y}(E T_0(\sigma)E_{-n}P^n) = \sigma(1)\Phi^J_{\infty,y}(P^n) + \Phi^J_{\infty,y}(E T_0(\sigma - \sigma(1))E_{-n}P^n).$$

Note that $\Phi^J_{\infty,y}(P^n) = \Phi^J_{\infty,y}(I - Q^n)\Phi^J_{\infty,y}(W_0(g))$ with $g$ continuous such that $g(y) = 1$, $g(\infty) = 0$, and so $\Phi^J_{\infty,y}(Q^nW_0(g)) = 0$ by Lemma 4.5.1 which means that

$$\Phi^J_{\infty,y}(P^n) = \Phi^J_{\infty,y}(I).$$

To prove that $\Phi^J_{\infty,y}(E T_0(\sigma - \sigma(1))E_{-n}) = 0$, multiply this coset by the coset $\Phi^J_{\infty,y}(P^nW_0(g)P^n)$ with $g$ a compactly supported continuous function which takes the value 1 at the point $y$. This last coset is the identity coset and applying De Boor’s estimates, we obtain by Theorem 4.1.3 that $E_{-n}P^nW_0(g)E_n = T_0(g^n_1)$, $g^n_1$ being a function in $C(\mathbb{T})^{d+1,d+1}$ which is zero away from the point 1, and whose characteristic function of the support tends strongly to zero as $n$ increases. Then $\|T_0(\sigma - \sigma(1))g^n_1\|$ also tends to zero as $n$ increases, which means that $\Phi^J_{\infty,y}(E T_0(\sigma - \sigma(1))E_{-n})$ is the coset $\Phi^J_{\infty,y}(0)$.

**Proposition 4.5.6.** The local algebra $\cal A^J_{\infty,y}$, $y \in \mathbb{R}$ is isomorphic to the algebra $\cal B = \text{alg}(I, P_1, \chi_+, W_0(\chi_+))$ and this isomorphism is given by $S_y$.

**Proof:** We have already that the homomorphisms $S_y$ are well defined in the quotient algebra $\cal A^J_{\infty,y}$ by Lemma 4.5.3. To prove that this homomorphism is really an isomorphism, define the following application,

$$S'_y : \cal B \to \cal A^J_{\infty,y}, \quad S'_y(A) = \Phi_{\infty,y}(U_yZ_{\gamma_n}AZ_{\gamma_n}^{-1}U_{-y}). \quad (4.19)$$

The application $S'_y$ is also an algebra homomorphism (note that $(U_yZ_{\gamma_n}AZ_{\gamma_n}^{-1}U_{-y}) \in \cal A$ for any $A \in \cal B$ as one can see by the generators of $\cal B$) and is not difficult to see that it is the inverse of $S_y$.

We are only left with the algebra at infinity.

### 4.6 The local algebra $\cal A^J_{\infty,\infty}$

It was not possible to find a useful identification of the local algebra $\cal A^J_{\infty,\infty}$ when considered as a whole, due to its complexity. But it is possible to identify meaningful unital subalgebras of it. First we will observe some properties.
Lemma 4.6.1. The local algebra \( \mathcal{A}^J_{\infty, \infty} \) is generated by the cosets \( \Phi^J_{\infty, \infty}(I) \), \( \Phi^J_{\infty, \infty}(\chi_+) \), \( \Phi^J_{\infty, \infty}(W^0(\chi_+)) \), \( \Phi^J_{\infty, \infty}(P_{\tau_0}) \) and \( \Phi^J_{\infty, \infty}(E_nT^0(\sigma)E_{-n}P^n + Q^n) \) with \( \sigma \in [C(T)]_{d+1,d+1} \).

Proof: The result is easily obtained from Lemmas 4.4.2 and 4.5.4. 

Proposition 4.6.2. The projection \( \Phi^J_{\infty, \infty}(\chi_+) \) belongs to the center of \( \mathcal{A}^J_{\infty, \infty} \).

Proof: First note that \( \chi_+ \) commutes with \( P_{\tau_0} \). The relation
\[
\Phi^J_{\infty, \infty}(W^0(\chi_+)) = \Phi^J_{\infty, \infty}(\chi_+W^0(\chi_+)),
\]
is proved as in Proposition 3.4.9. Finally \( \chi_+ \) commutes with \( Q^n \) and it is possible to write
\[
E_nT^0(\sigma)E_{-n}P^n\chi_+ - \chi_+ E_nT^0(\sigma)E_{-n}P^n = E_nT^0(\sigma)\hat{P}E_{-n}P^n - E_n\hat{P}T^0(\sigma)E_{-n}P^n = E_n(T^0(\sigma)\hat{P} - \hat{P}T^0(\sigma))E_{-n}P^n = E_nKE_{-n}P^n
\]
with \( K \) compact. To see that \( \Phi^J_{\infty, \infty}(E_nKE_{-n}P^n) = 0 \), approximate \( K \) by a finite dimensional operator \( K_0 \) and write
\[
\Phi^J_{\infty, \infty}(E_nK_0E_{-n}P^n) = \Phi^J_{\infty, \infty}(E_nK_0E_{-n}P^n)\Phi^J_{\infty, \infty}(f_\infty I) = \Phi^J_{\infty, \infty}(E_nK_0E_{-n}f_\infty P^n)
\]
for \( f_\infty \in C(\mathbb{R}) \) such that \( f_\infty(\infty) = 1 \) and \( f_\infty(x) = 0 \) for \( |x| < 1 \). Then for \( n \) large enough \( E_nK_0E_{-n}f_\infty P^n \) is the operator 0, and this yields our claim. 

Consider now the unital subalgebra \( \mathcal{A}^{J^1}_{\infty, \infty} \) of \( \mathcal{A}^J_{\infty, \infty} \) generated by the identity and the three projections \( \Phi^J_{\infty, \infty}(\chi_+) \), \( \Phi^J_{\infty, \infty}(W^0(\chi_+)) \) and \( \Phi^J_{\infty, \infty}(P^nP_{\tau_0}) \). As it was seen in Lemma 4.6.2 above, this algebra has a non trivial center, because the coset \( \Phi^J_{\infty, \infty}(\chi_+) \) commutes with the other generators. We can localize, obtaining algebras \( \mathcal{A}^{J^1}_{\infty, \infty, \pm} \), which are generated by two projections (plus the identity) and obtaining also the canonical homomorphism \( \Phi^{J^1}_{\infty, \infty, \pm} \) from \( \mathcal{A} \) onto \( \mathcal{A}^{J^1}_{\infty, \infty, \pm} \). We will introduce now the applications that to a sequence \( (A_n) \) of \( \mathcal{A} \) associate
\[
S^{1, \pm}_{\infty n}(A_n) = \hat{V}_{\pm n}A_n\hat{V}_{\pm n},
\]
and the strong limits
\[
S^{1, \pm}_{\infty}(A_n) = s-lim_{n \to \infty} S^{1, \pm}_{\infty n}(A_n)
\]
if they exist.

Lemma 4.6.3. The strong limits \( S^{1, \pm}_{\infty}(A_n) \) exist for all \( (A_n) \in \mathcal{A} \). In particular,

(a) \( S^{1, \pm}_{\infty}(\chi_+) = I \)
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(b) $S_{1}^{\pm}(P_{\tau_n}) = \chi_{\pm}$, $S_{1}^{\pm}(E_n T^{0}(\sigma) E_{-n} P^n + Q^n) = \sigma(1)I$ for $\sigma \in [C(\mathbb{T})]_{d+1,d+1}$;

c) $S_{1}^{\pm}(cI) = c(\pm \infty)I$ for $c \in PC(\mathbb{R})$;

d) $S_{1}^{\pm}(W^{0}(a)) = W^{0}(a)$ for $a \in PC(\mathbb{R})$;

e) if $(j_n) \in \mathcal{A} \cap \mathcal{J}$, then $S_{1}^{\pm}(j_n) \in \mathcal{K}$.

Proof: The only statement that is not proved in Lemma 3.4.10 is the second part of (b). For this it is only necessary to remark that $\tilde{V}_{\tau_n}Q^nV_{\tau_n} = Q^n$ and that $\tilde{V}_{\tau_n}E_n T^{0}(\sigma) E_{-n} P^n \tilde{V}_{\tau_n}$ is as well equal to $E_n T^{0}(\sigma) E_{-n} P^n$ because the operators $E_{-n} \tilde{V}_{\tau_n}E_n$ are still translation operators in $l_{d+1}$.

Using these homomorphisms (due to its construction and Lemma 4.6.3 it is not difficult to see that $S_{1}^{\pm}$ are indeed homomorphisms) and using the procedure described in the last chapter, one can see that

$$\sigma_{\mathcal{A}_{\infty,\infty}^{J}_{1}}(\Phi_{\infty,\infty}^{J}(P^n P_{\tau_n} W^{0}(\chi_{\pm}) P^n P_{\tau_n})) = [0,1]$$

and so it is possible to apply the Two Projections theorem. This theorem gives us symbol mappings $N^{\pm}$, to a space of matrix functions,

$$N^{\pm}(\Phi_{\infty,\infty}^{J}(\chi_{\pm})) = e^{'}: t \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad (4.23)$$

$$N^{\pm}(\Phi_{\infty,\infty}^{J}(P^n P_{\tau_n})) = p_{1}^{'}: t \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad (4.24)$$

$$N^{\pm}(\Phi_{\infty,\infty}^{J}(W^{0}(\chi_{\pm}))) = p_{2}^{'}: t \mapsto \begin{bmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{bmatrix}; \quad (4.25)$$

with $t \in [0,1]$.

So define the homomorphism,

$$M^{\pm} = N^{\pm}\Phi_{\infty,\infty}^{J}.$$

We just proved the following result:

**Proposition 4.6.4.** The local algebras $\mathcal{A}_{\infty,\infty}^{J}_{1}$ are isomorphic to the unital $C^*$-subalgebra of the algebra of 2 by 2 matrix functions defined on $[0,1]$, which are diagonal at $\{0,1\}$ (and which is generated by $e^{'}$, $p_{1}^{'}$ and $p_{2}^{'}$), and these isomorphisms are given by $N^{\pm}$. The coset $\Phi_{\infty,\infty}^{J}(A_n) \in \mathcal{A}_{\infty,\infty}^{J}_{1}$ is invertible in the local algebra $\mathcal{A}_{\infty,\infty}^{J}$ if and
only if $M^\pm(A_n)$ are invertible. In particular

$$M^\pm(cI) : t \mapsto \begin{bmatrix} c(\pm \infty) & 0 \\ 0 & c(\pm \infty) \end{bmatrix};$$

$$M^\pm(P^n P_{\tau_n}) : t \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};$$

$$M^\pm(W^0(a)) : t \mapsto a(-\infty) \begin{bmatrix} 1-t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & t \end{bmatrix} + a(+\infty) \begin{bmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{bmatrix}.$$  

\[ 4.6.1 \text{ Remark} \] We could have chosen instead of the coset $\Phi^J_{\infty,\infty}(P^n P_{\tau_n})$ as generator of $A^T_{\infty,\infty,\pm}$, the coset $\Phi^J_{\infty,\infty}(P_{\tau_n})$ or the coset $\Phi^J_{\infty,\infty}(P^n)$. In this case we would be studying the pure finite section case (as was done in the last chapter) or the infinite Galerkin method (see also \[ 30 \]).

Another unital subalgebra of $A^T_{\infty,\infty}$ that is possible to study is the algebra $A^T_{\infty,\infty}$ generated by the identity and the sequences $\Phi^J_{\infty,\infty}(\chi_+)$, $\Phi^J_{\infty,\infty}(P_{\tau_n})$, $\Phi^J_{\infty,\infty}(P^n W^0(\chi_+)(P^n + Q^n)$ and $\Phi^J_{\infty,\infty}(E_n T^n(S) E_{-n} P^n + Q^n)$. As before, this algebra has a non trivial center, because the coset $\Phi^J_{\infty,\infty}(\chi_+)$ commutes with the other generators. We can localize, using ideals $I_{\infty,\infty,\pm}$, obtaining algebras $A^T_{\infty,\infty,\pm}$ and the canonical homomorphisms $\Phi^J_{\infty,\infty,\pm}$ from $A$ onto $A^T_{\infty,\infty,\pm}$. If we write the space $L^2(\mathbb{R})$ as the direct sum

$$L^2(\mathbb{R}) = \text{Im}(P^n) \oplus \text{Im}(Q^n) \quad (4.27)$$

then each sequence in $A$ is isomorphic to a matrix sequence

$$\begin{bmatrix} P^n A_n P^n & P^n A_n Q^n \\ Q^n A_n P^n & Q^n A_n Q^n \end{bmatrix}. \quad (4.28)$$

We denote this isomorphism by $\Theta_n$.

In order to find an isomorphism that will permit to identify this algebra define the application

$$\tilde{E}_n : l^2_{d+1} \oplus (S^n)^{\perp} \rightarrow S^n \oplus (S^n)^{\perp}, \quad \tilde{E}_n = \begin{bmatrix} E_n & 0 \\ 0 & Q^n \end{bmatrix} \quad (4.29)$$

with inverse

$$\tilde{E}_{-n} : S^n \oplus (S^n)^{\perp} \rightarrow l^2_{d+1} \oplus (S^n)^{\perp}, \quad \tilde{E}_{-n} = \begin{bmatrix} E_{-n} & 0 \\ 0 & Q^n \end{bmatrix}. \quad (4.30)$$

Using $\tilde{E}_n$ and $\tilde{E}_{-n}$ it is possible to define the applications that to a sequence $(A_n)$ of $A$ associate

$$S^2_{\infty,\infty}(A_n) = \tilde{E}_{-n} \Theta_n \tilde{V}_{\pm \tau_n} A_n \tilde{V}_{\mp \tau_n} \Theta_n^{-1} \tilde{E}_n, \quad (4.31)$$

and the strong limits

$$S^2_{\infty}(A_n) = \text{s-lim} S^2_{\infty,\infty}(A_n) \quad (4.32)$$

if they exist. Note that the strong limit is a matrix operator acting on $L^2(\mathbb{R}) \oplus \{0\}$, so that it is possible to ignore the trivial part and consider it as a scalar operator.
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**Lemma 4.6.5.** The strong limits $S^{2\pm}_\infty(A_n)$ exist for all $(A_n) \in A$. In particular,

(a) $S^{2\pm}_\infty(E_n T^0(\sigma) E_{-n} P^n + Q^n) = T^0(\sigma)$ for $\sigma \in [C(\mathbb{T})]_1^{d+1,d+1}$;

(b) $S^{2\pm}_\infty(c) = c(\pm \infty) I$ for $c \in PC_Z(\mathbb{R})$;

(c) $S^{2\pm}_\infty(P_{\tau_n}) = I - \hat{P}$, $S^{2-}_\infty(P_{\tau_n}) = \hat{P}$;

(d) $S^{2\pm}_\infty(W^0(a)) = T^0(-\infty \frac{1-\sigma_0}{2} + a(\pm \infty \frac{1+\sigma_0}{2})$ for $a \in PC(\mathbb{R})$;

(e) if $(j_n) \in J$, then $S^{2\pm}_\infty(j_n) = 0$.

**Proof:** All strong limits except $S^{2\pm}_\infty(P_{\tau_n})$ are proved in [30, Proposition 16]. For these note that $\hat{V}_{+\tau_n} P_{\tau_n} \hat{V}_{-\tau_n} = \chi_{[0,2\tau_n]}$ and $\hat{V}_{-\tau_n} P_{\tau_n} \hat{V}_{+\tau_n} = \chi_{[-2\tau_n,0]}$.

To prove the last assertion note again that for $G_n \in G$, $\|S^{2\pm}_\infty(G_n)\| \leq \|G_n\| \to 0$. For $(j_n) \in J$, note that $P^n \hat{V}_{+\tau_n} E_n, P^n R_n \hat{V}_{+\tau_n} E_n$ and $P^n \hat{V}_{-\tau_n} E_n$ tend weakly to zero as $n \to \infty$, and $E_{-n} \hat{V}_{+\tau_n} P^n, E_{-n} \hat{V}_{+\tau_n} R_n P^n$ and $E_{-n} \hat{V}_{-\tau_n} \hat{V}_{+\tau_n} P^n$ are uniformly bounded.

To identify the local algebras $A^{J2}_{\infty,\infty,\pm}$ we will use the strong limits $S^{2\pm}_\infty$, which will be seen to be well defined homomorphisms. We start by remarking that, by Lemma 4.6.5, $S^{2\pm}_\infty(J) = 0$, $S^{2\pm}_\infty(I_\infty) = 0$ and $S^{2\pm}_\infty(I_{\infty,\infty}) = 0$. As by the same lemma $S^{2\pm}_\infty(I_{\infty,\infty,\pm}) = 0$ respectively then the homomorphisms $S^{2\pm}_\infty$ are well defined from $A^{J2}_{\infty,\infty,\pm}$ into the subalgebra $D$ of $L(l^2)$ generated by the projection $\hat{P}$, the Laurent operators $T^0(\sigma)$ with $\sigma \in [C(\mathbb{T})]_1^{d+1,d+1}$ and $T^0(\sigma_P)$ (i.e. $D$ is generated by $\hat{P}$ and the Laurent operators $T^0(a)$, with $a \in PC(\mathbb{T})$ such that the only point of discontinuity possible is $t = 1$).

Using these homomorphisms it is then possible to prove the following result.

**Proposition 4.6.6.** The local algebras $A^{J}_{\infty,\infty,\pm}$ are isomorphic to the algebra $D$ and the isomorphisms are given by $S^{2\pm}_\infty$.

**Proof:** We just have to find the left inverse homomorphism. Let $S^{\pm J}_\infty: D \to A^{J}_{\infty,\infty,\pm}$ be defined as

$$S^{\pm J}_\infty(A) = \Phi^{J}_{\infty,\infty,\pm}(\hat{V}_{\pm\tau_n} E_n AE_{-n} P^n \hat{V}_{\mp\tau_n} + Q^n).$$

These two homomorphisms are well defined because $(\hat{V}_{\pm\tau_n} E_n AE_{-n} P^n \hat{V}_{\mp} + Q^n) \in A$ for all operators $A \in D$ as can be seen by the generators of $D$. And as

$$\Phi^{J}_{\infty,\infty,\pm}(A_n) = S^{\pm J}_\infty(S^{2\pm}_\infty(A_n))$$

for all $A_n \in A$, the result is proved.
4.7 The Stability Theorem

We resume in the next theorem the results obtained.

**Theorem 4.7.1 (Stability).** Let \((A_n)\) be a sequence

(a) belonging to the algebra generated by the projection sequence \((P_{\tau n} P^n)\) (or \((P_{\tau n})\), or \((P^n))\), the constant sequences \((cI)\) with \(c \in PC_{Z}(\hat{R})\) and the constant sequences \((W^0(a))\) with \(a \in PC(\hat{R})\). Then \((A_n)\) is stable if and only if the operators \(O^0((A_n)), O^1((A_n)), O_x(A_n), S_y(A_n)\) and \(M^\pm(A_n)\) are invertible for \(x, y \in \mathbb{R}\).

(b) belonging to the algebra generated by the finite section projection sequence \((P_{\tau n})\), the sequences \((E_n \Sigma E_n + Q^n)\) with \(\Sigma\) continuous and such that \(\Sigma(1)\) is equal to the identity matrix times a constant, the sequences \((P^n W^0(a) P^n + Q^n)\) with \(a \in PC(\hat{R})\) and the constant sequences \((cI)\) with \(c \in PC_{Z}(\hat{R})\). Then \((A_n)\) is stable if and only if the operators \(O^0((A_n)), O^1((A_n)), O_x(A_n), S_y(A_n)\) and \(S^2_{\infty}(A_n)\) are invertible for \(x, y \in \mathbb{R}\).

4.8 Examples and applications

As in the last chapter, we will give now some examples of application of the Stability Theorem to specific operators and methods.

4.8.1 The finite Galerkin method

Consider the approximation method

\[
(P^n P_{\tau n} A P^n P_{\tau n} + I - P^n P_{\tau n}) u_n = v
\]

(4.34)

to solve the equation

\[
Au = v \quad u, v \in L^2(\mathbb{R}),
\]

with \(u_n \in L^2(\mathbb{R})\).

Singular Integral Operators

Consider now the operator

\[
A = c_1 W^0(\chi_+) + c_2 W^0(\chi_-)
\]

(4.35)

with \(c_{1,2} \in PC_{Z}(\hat{R})\). Applying the Stability Theorem to the operator \(A\) we obtain the following result:
Corollary 4.8.1. The approximation method (4.34) applies to $A$ if and only if the operator $A$, the operator
\[
P_1((c_1(-\infty)\chi_+ + c_1(+\infty)\chi_-)W^0(\chi_+) + (c_2(-\infty)\chi_- + c_2(+\infty)\chi_+)W^0(\chi_-))P_1 + Q_1,
\]
for $x \in \mathbb{Z}$ the operators
\[
P^1((c_1(x^-)\chi_- + c_1(x^+)\chi_+)W^0(\chi_+) + (c_2(x^-)\chi_- + c_2(x^+)\chi_+)W^0(\chi_-))P^1 + Q^1
\]
and for $x \in \mathbb{R} \setminus \mathbb{Z}$,
\[
P^1(c_1(x)W^0(\chi_+) + c_2(x)W^0(\chi_-))P^1 + Q^1
\]
are invertible.

Proof: The direct use of Theorem 4.7.1 leads to the following operators (resp. operator functions) that must be invertible, in order to guarantee the stability of the sequence $A_n = (P^n P_{\tau_n} A P^n P_{\tau_n} + I - P^n P_{\tau_n})$:

(a) $A = c_1W^0(\chi_+) + c_2W^0(\chi_-);

(b) $A_{11} = ((c_1(-\infty)\chi_- + c_1(+\infty)\chi_+)W^0(\chi_-) + (c_2(-\infty)\chi_- + c_2(+\infty)\chi_+)W^0(\chi_+);

(c) $O_x(A_n) = Q^1 +
\[
P^1((c_1(x^-)\chi_- + c_1(x^+)\chi_+)W^0(\chi_+) + (c_2(x^-)\chi_- + c_2(x^+)\chi_+)W^0(\chi_-))P^1,
\]
for $x \in \mathbb{R};

(d) $S_0(A_n) = Q_1 +
\[
P_1((c_1(-\infty)\chi_- + c_1(+\infty)\chi_+)W^0(\chi_+) + (c_2(-\infty)\chi_- + c_2(+\infty)\chi_+)W^0(\chi_-))P_1;
\]

(e) $S_y(A_n) = \begin{cases} P_1(c_2(-\infty)\chi_- + c_2(+\infty)\chi_+)P_1 + Q_1 & y < 0 \\ P_1(c_1(-\infty)\chi_- + c_1(+\infty)\chi_+)P_1 + Q_1 & y > 0 \end{cases};

(f) $M^\pm(A_n) : x \mapsto \begin{bmatrix} c_1(\pm\infty)(1-x) + c_2(\pm\infty)x & 0 \\ 0 & 1 \end{bmatrix}$, for $x \in [0,1].$

The fact that $A$ must be invertible implies immediately the invertibility of $\tilde{A} (= A_{11})$ and that $c_{1,2}(\pm\infty)$ are different from zero, which also means that $S_y$ is invertible for $y \in \mathbb{R} \setminus \{0\}$. If $x$ is non-integer, $O_x(A_n)$ simplifies to $P^1(c_1(x)W^0(\chi_+) + c_2(x)W^0(\chi_-))P^1 + Q^1$. Finally one can see that the invertibility of $S_0(A_n)$ implies the invertibility of $M^\pm(A_n)$.

Note that the operators (4.37) and (4.38) by the use of $E_1$ and $E_{-1}$ can be reduced to operators acting on $\ell^2_{d+1}$, and by studying the characteristics of the generating functions
of the Laurent operators it is possible to extract verifiable invertibility conditions. Unfortunately, for \( d > 0 \) these functions are not completely studied (see the collocation method subsection, where some details are given for \( d = 1 \)).

Let \( A \) be now the operator

\[
A = W^0(a) \chi_+ + W^0(b) \chi_-
\]

with \( a, b \in PC(\mathbb{R}) \). The following result is then obtained:

**Corollary 4.8.2.** The approximation method \( (4.34) \) applies to \( A \) (given by \( 4.39 \)) if and only if the following conditions are satisfied:

(a) The operator \( A \) is invertible;

(b) The operator \( \chi_+ W^0(\tilde{a}) \chi_+ + \chi_- W^0(\tilde{b}) \chi_- \) is invertible;

(c) The operator \( Q^1 + P^1 \left( (a(+\infty)\chi_+ + b(+\infty)\chi_-) W^0(\chi_+) + (a(-\infty)\chi_+ + b(-\infty)\chi_-) W^0(\chi_-) \right) P^1 \)

is invertible;

(d) The operators \( P^1(b(-\infty)W^0(\chi_-) + b(+\infty)W^0(\chi_+))P^1 + Q^1 \) and \( P^1(a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+))P^1 + Q^1 \) are invertible.

(e) The operators

\[
P_1 \left( W^0(\chi_-)(a(y^-)\chi_+ + b(y^-)\chi_-) + W^0(\chi_+)(a(y^+)\chi_+ + b(y^+)\chi_-) \right) P_1 + Q_1
\]

are invertible for \( y \in \mathbb{R} \);

**Proof:** A necessary and sufficient condition for the stability of the approximation sequence is the invertibility of the following operators (resp. operator functions):

(i) \( A = W^0(a) \chi_+ + W^0(b) \chi_- \);

(ii) \( A_{11} = \chi_+ W^0(\tilde{a}) \chi_+ + \chi_- W^0(\tilde{b}) \chi_- \);

(iii) \( O_0(A_n) = Q^1 + P^1 \left( (a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+)) \chi_+ + \right.

\[
(b(-\infty)W^0(\chi_-) + b(+\infty)W^0(\chi_+)) \chi_- \right) P^1;
\]

(iv) \( O_x(A_n) = \begin{cases} P^1(b(-\infty)W^0(\chi_-) + b(+\infty)W^0(\chi_+))P^1 + Q^1 & x < 0 \\ P^1(a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+))P^1 + Q^1 & x > 0 \end{cases} \)
4.8. EXAMPLES AND APPLICATIONS

(v) \( S_y(A_n) = Q_1 + \)

\[
P_1 \left( W^0(\chi_+)(a(y^-)\chi_+ + b(y^-)\chi_-) + W^0(\chi_+)(a(y^+)\chi_+ + b(y^+)\chi_-) \right) P_1
\]

for \( y \in \mathbb{R} \);

(vi) \( M^+(A_n) : x \mapsto \begin{bmatrix} a(-\infty)(1-x) + a(+\infty)x & 0 \\ 0 & 1 \end{bmatrix}, x \in [0,1]. \)

(vii) \( M^-(A_n) : x \mapsto \begin{bmatrix} b(-\infty)(1-x) + b(+\infty)x & 0 \\ 0 & 1 \end{bmatrix}, x \in [0,1]. \)

The first to the fifth conditions of the Corollary come directly from (i)-(v) and the invertibility of \( A \) implies the invertibility of \( M^\pm(A_n) \).

Products of convolution and multiplication operators

Now let \( A \) be the operator

\[
A = \sum_{k=1}^{n} \prod_{l=1}^{m} c_{kl}^{(1)} W^0(a_{kl}) c_{kl}^{(2)}
\]

with \( a_{kl} \in PC(\mathbb{R}) \) and \( c_{kl}^{(1)}, c_{kl}^{(2)} \in PC_\mathbb{Z}(\mathbb{R}) \). With the choice of the functions \( a_{kl} \) and \( c_{kl}^{(1)}, c_{kl}^{(2)} \) it is possible to obtain Wiener-Hopf type operators \( W(a) = \chi_+ W^0(a) \chi_+ + \chi_- \) as well as products of these operators.

**Corollary 4.8.3.** The approximation method (4.34) applies to \( A \) if and only if the following operators (resp. operator functions) are invertible:

(a) The operator \( A \);

(b) The operator

\[
\sum_{k=1}^{n} \prod_{l=1}^{m} \left( c_{kl}^{(1)}(\chi_- W^0(\tilde{a}_{kl}) & c_{kl}^{(2)}(-\infty) + c_{kl}^{(1)}(+\infty) \chi_+ W^0(\tilde{a}_{kl}) \chi_+ c_{kl}^{(2)}(+\infty) \right)
\]

(c) For \( x \in \mathbb{R} \),

\[
P_1 \left( \sum_{k=1}^{n} \prod_{l=1}^{m} \left( c_{kl}^{(1)}(x^-) \chi_- + c_{kl}^{(1)}(x^+) \chi_+ \right) \left( a_{kl}(-\infty) W^0(\chi_-) + a_{kl}(+\infty) W^0(\chi_+) \right) \right) P_1 + Q_1;
\]
(d) For \( y \in \mathbb{R} \),
\[
P_1 \left( \sum_{k=1}^{a} \prod_{l=1}^{m_n} \left( c_{kl}^{(1)}(-\infty) \chi_- + c_{kl}^{(1)}(+\infty) \chi_+ \right) (a_{kl}(y^-) W^0(\chi_-) + a_{kl}(y^+) W^0(\chi_+)) \right) P_1 + Q_1;
\]
\[
t \mapsto \sum_{k=1}^{n} \prod_{l=1}^{m_n} \left( c_{kl}^{(1)}(-\infty) (a_{kl}(-\infty) (1 - t) + a_{kl}(+\infty) t) c_{kl}^{(2)} (-\infty) \right), \quad t \in [0, 1]
\]
\[
(4.41)
\]

Proof: Apply the Stability Theorem.

Specifying the operator \( A \) to be the Wiener-Hopf operator
\[
W(a) = \chi_+ W^0(a) \chi_+ + \chi_-
\]
we obtain the following result:

**Corollary 4.8.4.** Let \( a \in \text{PC}(\mathbb{R}) \). Then the approximation method \([4.34]\) applies to \( A \) if and only if the operator \( W(a) \) is invertible.

Proof: We refer to Corollary 4.8.3. From (a) we obtain that \( W(a) \) must be invertible. From (b) that \( W(\tilde{a}) = \chi_+ W^0(\tilde{a}) \chi_+ + \chi_- \) must also be invertible. But it is not too difficult to verify that the invertibility of \( W(a) \) is equivalent to the invertibility of \( W(\tilde{a}) \).

Regarding (c), the only non-trivial case appears when \( x = 0 \). At that point we obtain the operator
\[
P_1 \chi_+ (a(-\infty) W^0(\chi_-) + a(+\infty) W^0(\chi_+)) \chi_+ P_1 + \chi_- P_1 + Q_1.
\]
This operator, by the use of the operators \( E_1 \) and \( E_{-1} \) (defined in \([4.5]\) and \([4.6]\)), can be reduced to an operator acting on \( l^2 \) and using \([17] \) Proposition 2.6 one can see that the operator is invertible if and only if the segment joining the point \( a(-\infty) \) to the point \( a(+\infty) \) on the complex plane does not contain the point \( 0 \). And this is true if the operator \( W(a) \) is Fredholm.

From (d) it is obtained that for \( y \in \mathbb{R} \) the operators
\[
P_1 \chi_+ (a(y^-) W^0(\chi_-) + a(y^+) W^0(\chi_+)) \chi_+ P_1 + \chi_- P_1 + Q_1
\]
are invertible. But the above operator is nothing else than a singular integral type operator acting in the interval \([0, 1]\) and is invertible if and only if the segment joining the point \( a(y^-) \) to the point \( a(y^+) \) on the complex plane does not contain the point 0 (see \([22] \) Chapter IV). But this is again a consequence of the Fredholmness of \( W(a) \).

Finally, (e) is trivially verified, and (f) is implied by (c).
Remark 4.8.1. Note that some operators appearing in the corollaries can be simplified (and verifiable invertibility conditions extracted) if we have scalar operators \( A \) (which is the case we are specifically treating). For example, consider the operator (4.36). By writing it in the form

\[
a'(W^0(X_+)) + W^0(X_-), \quad a'(x) = \begin{cases} \frac{c_1(-\infty)}{c_2(-\infty)} & 0 < x < 1 \\ \frac{c_1(+\infty)}{c_2(+\infty)} & -1 < x < 0 \\ 1 & |x| > 1 \end{cases},
\]

and applying the usual invertibility conditions for singular integral operators (see [22]), it is possible to conclude that the operator is invertible if and only if for any \( s \in \mathbb{R} \) the point 0 is not contained in the area limited by the triangle defined by the points \( \{1, \frac{c_1(-\infty)}{c_2(-\infty)}, \frac{c_1(+\infty)}{c_2(+\infty)}\} \). But for the results to be also true in the non-scalar cases (see the final remarks) we opted to present them in this form.

### 4.8.2 The finite collocation method

It is necessary to verify first that the sequence algebra \( \mathcal{A} \), contains indeed sequences corresponding to collocation methods. As a simple example, put \( d = 0 \) and consider the collocation projection

\[
(L^n u)(x) = \sum_{j=-\infty}^{+\infty} u \left( j + \frac{1}{2} \right) \chi_{[\frac{j}{n}, \frac{j+1}{n}]}(x).
\]  

(4.42)

By Theorem 2.15 in [17], the operator \( E_{-n}L^nS_{\mathbb{R}}E_n : l^2 \to l^2 \) is independent of \( n \) and equal to \( T^0(\sigma_L) \), with \( \sigma_L \) a piecewise continuous function on \( \mathbb{T} \) which is continuous at all points except the point 1, and verifies at this point the equation

\[
\sigma_L(1+) - \sigma_L(1-) = -2.
\]

Remembering that \( \sigma_P \) is the function corresponding to the sequence \( E_{-n}P^nS_{\mathbb{R}}E_n \), by Theorem 4.1.2 we see that \( \sigma_L - \sigma_P \) is a continuous function in \( \mathbb{T} \) which takes the value 0 at the point 1. So the sequence \( (L^nS_{\mathbb{R}}P^n) \in \mathcal{A} \). That the sequences \( (L^nW^0(a)P^n) \) with \( a \in PC(\mathbb{R}) \) are also in \( \mathcal{A} \), was our aim to prove. This would follow in a natural way if the following conjecture was true:

**Conjecture.** For any fixed \( s \in \mathbb{R} \), \( L^nS_{\mathbb{R}}U_sP^n \) is uniformly bounded in \( L^2(\mathbb{R}) \) and \( \|L^nS_{\mathbb{R}}(I - P^n)U_sP^n\| \to 0 \) as \( n \to \infty \).

Unfortunately it was not possible to prove this conjecture, and it is only possible to prove that the collocation method for the singular integral operator is included. But for the multiplication operators, it was possible to prove the following results. We will start with a definition.
Definition 4.8.1. A piecewise continuous function $c$ will be called uniformly piecewise continuous if it satisfies the condition
\[
\forall \epsilon > 0 \exists \delta > 0 \forall k \in \mathbb{Z} \forall x, y \in [k, k+1] |x - y| < \delta \Rightarrow |c(x) - c(y)| < \epsilon. \tag{4.43}
\]
The module of continuity $\omega'(c, \delta)$ is defined as
\[
\omega'(c, \delta) = \sup_{k \in \mathbb{Z}} \sup_{x, y \in [k, k+1], |x - y| < \delta} |c(x) - c(y)|
\]
and tends to zero as $\delta$ goes to zero.

Lemma 4.8.5. Let $c$ belong to $PC_{\mathbb{Z}}(\mathbb{R})$. Then $c$ is piecewise uniformly continuous.

Proof: As the function $c$ is piecewise continuous and its restriction to each open interval $[k, k+1]$ is continuous, it can be continuously extended to the closed interval $[k, k+1]$ and so it must be uniformly continuous in $[k, k+1]$. Given an $\epsilon > 0$, due to the piecewise continuity at infinity it is possible to find a $k_0$ such that for any $x > k_0$ one has $|c(x) - c(+\infty)| < \epsilon/2$ and for any $x < -k_0$ one has $|c(x) - c(-\infty)| < \epsilon/2$. Then put $\delta = \min_{|k| \leq k_0} \delta_k$ where $\delta_k$ are positive values corresponding to $\epsilon$ by the uniform continuity in each interval $[k, k+1]$.

Lemma 4.8.6. Let $f \in L^\infty(\mathbb{R})$ such that the set of points in which $f$ has discontinuities is contained in $\mathbb{Z}$. Then $L^n f P^n$ is uniformly bounded in $L^2(\mathbb{R})$. If $f$ is piecewise uniformly continuous then the norm $\|(I - L^n) f P^n\|_{L^2}$ tends to zero as $n \to \infty$.

Proof: Given $u \in L^2(\mathbb{R})$, one can write directly the norm of $L^n f P^n u$ (squared) as
\[
\|L^n f P^n u\|_2^2 = \int_{-\infty}^{+\infty} \left| \sum_j f\left(\frac{2j + 1}{2n}\right) n \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) dt \chi_{[\frac{j}{n}, \frac{j+1}{n}]}(x) \right|^2 dx =
\]
\[
= \sum_j \int_{\frac{j}{n}}^{\frac{j+1}{n}} \left| f\left(\frac{2j + 1}{2n}\right) n \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) dt \right|^2 dx = \frac{1}{n} \sum_j \left| f\left(\frac{2j + 1}{2n}\right) n \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) dt \right|^2 \leq
\]
\[
\leq \|f\|^2 \frac{1}{n} \sum_j \left| n \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) dt \right|^2 \leq \|f\|_{\infty}^2 \|u\|_2^2. \tag{4.45}
\]
the last step by applying the Cauchy-Schwartz inequality. For the second part write
\[
\|(I - L^n) f P^n u\|_2^2 =
\]
\[
= \left\| \sum_j \left( f\left(\frac{2j + 1}{2n}\right) n \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) dt \chi_{[\frac{j}{n}, \frac{j+1}{n}]}(x) - f(x) n \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) dt \chi_{[\frac{j}{n}, \frac{j+1}{n}]}(x) \right) \right\|_2^2 =
\]
\[
= \int_{-\infty}^{+\infty} \left| \sum_j \left( f\left(\frac{2j + 1}{2n}\right) - f(x) \right) n \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) dt \chi_{[\frac{j}{n}, \frac{j+1}{n}]}(x) \right|^2 dx \leq
\]
\[
\leq \sum_j \int_{\frac{j}{n}}^{\frac{j+1}{n}} \left| f\left(\frac{2j + 1}{2n}\right) - f(x) \right|^2 \leq \frac{1}{n} \sum_j \int_{\frac{j}{n}}^{\frac{j+1}{n}} \left| n \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) dt \right|^2 dx. \tag{4.46}
\]
By applying the Cauchy-Schwartz inequality we obtain
\[
\sum_j \int_{\frac{j}{n}}^{\frac{j+1}{n}} \left| f\left(\frac{2j+1}{2n}\right) - f(x) \right|^2 \left| \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) \, dt \right|^2 \, dx \leq \sum_j \int_{\frac{j}{n}}^{\frac{j+1}{n}} \left| f\left(\frac{2j+1}{2n}\right) - f(x) \right|^2 n \left| \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) \, dt \right|^2 \, dx \leq \omega'(f, \frac{1}{2n}) \leq \omega'(f, \frac{1}{2n})^2 \left\| u \right\|_2^2. \tag{4.47}
\]

with \( \omega'(f, \delta) \) defined in (4.44) and because \( \omega'(f, \frac{1}{2n}) \) tends to zero as \( n \to \infty \) (\( f \) being uniformly piecewise continuous) the result follows.

**Lemma 4.8.7.** Let \( f \in PC_{\mathbb{Z}}(\mathbb{R}) \). Then the norm \( \|(I - P^n)fP^n\|_{L^2} \) tends to zero as \( n \to \infty \).

**Proof:** Consider \( f \) to be a positive function. As any complex function can be written in the form \( f = f^+ - f^- + i(f_1 - f_2) \) with \( f^\pm \) and \( f_1, f_2 \) positive functions there is no loss of generality. Writing the norm explicitly we obtain
\[
\|(I - L^n)fP^n\|_2^2 = \sum_j \left( n \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(t) \, dt \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) \, dt \chi_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}(x) - f(x) \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(t) \, dt \chi_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}(x) \right)^2. \tag{4.48}
\]

By the Mean Value theorem we have that \( n \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(t) \, dt = f(x_j) \), with \( x_j \in [\frac{j}{n}, \frac{j+1}{n}] \).

The rest of the proof follows now the same line as the proof of the last lemma.

Note that the two above results together give in particular the following important corollary, whose proof is trivial:

**Corollary 4.8.8.** If \( f \in PC_{\mathbb{Z}}(\mathbb{R}) \), then \( \|(L^n - P^n)fP^n\| \to 0 \), that is, the collocation method can be considered here as a perturbation (going to zero in the operator norm) of the Galerkin method.

The following simple result is also of importance.

**Lemma 4.8.9.** Let \( f \in L^\infty \) such that the set of points in which \( f \) is discontinuous is contained in \( \mathbb{Z} \) and \( L^n u \) belong to \( L^2(\mathbb{R}) \). Then \( L^n f u = L^n f P^n L^n u \).

**Proof:** First note that \( P^n L^n = L^n \). Then the result comes directly by writing the operator definitions.
Let \( R^2(\mathbb{R}) \) be the closure of the linear space spanned by the characteristic functions \( \chi_{[a,b]} \), with \(-\infty < a < b < +\infty\), in the norm
\[
\|v\|_{R^2} = \|v\|_2 + \left( \sum_{l} \sup_{x \in [l,l+1]} |v(x)|^2 \right)^{\frac{1}{2}}.
\]
(4.49)

This space contains for example all continuous functions with compact support and all piecewise continuous functions with compact support and one-sided limits from the right at all points (see [17, Section 2.7]).

The collocation projection \( L^n \) is well defined in this space and so it is possible to consider the approximation method
\[
(L^n P_{r_n} A P^n P_{r_n} + P^n Q_{r_n} P_{r_n} + P_{r_n} Q) u_n = L^n v
\]
(4.50)
to solve the equation
\[
Au = v \quad u \in L^2(\mathbb{R}), \ v \in R^2(\mathbb{R}),
\]
with \( u_n \in S^n \).

Let
\[
A = c_1 W^0(\chi_+) + c_2 W^0(\chi_-)
\]
(4.51)
with \( c_{1,2} \in PC_{Z}(\mathbb{R}) \).

**Corollary 4.8.10.** The approximation method (4.50) applies to \( A \) if and only if the operator \( A \), the operator
\[
P_1((c_1(-\infty)\chi_+ + c_1(+\infty)\chi_+)W^0(\chi_+) + (c_2(-\infty)\chi_- + c_2(+\infty)\chi_+)W^0(\chi_-))P_1 + Q_1,
\]
(4.52)
for \( x \in Z \) the operators
\[
L^1((c_1(x^-)\chi_+ + c_1(x^+)\chi_+)W^0(\chi_+) + (c_2(x^-)\chi_- + c_2(x^+)\chi_+)W^0(\chi_-))P^1 + Q^1
\]
(4.53)
and for \( x \in \mathbb{R} \setminus Z \) the operators,
\[
L^1(c_1(x)W^0(\chi_+) + c_2(x)W^0(\chi_-))P^1 + Q^1
\]
(4.54)
are invertible as well as the operators in \( l^2_{d+1} \),
\[
\hat{Q}T^0 \left( c_1(+\infty) \frac{1 + \sigma_L}{2} + c_2(+\infty) \frac{1 - \sigma_L}{2} \right) \hat{Q} + \hat{P}
\]
(4.55)
\[
\hat{P}T^0 \left( c_1(-\infty) \frac{1 + \sigma_L}{2} + c_2(-\infty) \frac{1 - \sigma_L}{2} \right) \hat{P} + \hat{Q}.
\]
(4.56)

**Proof:** The first thing it is necessary to verify is that the sequence
\[
(L^n P_{r_n} A P^n P_{r_n} + P^n Q_{r_n} + Q^n)
\]
belongs to our algebra of approximating sequences.
4.8. EXAMPLES AND APPLICATIONS

But as \( W^0(\chi_\pm) = (I \pm S_{\mathbb{R}})/2 \), by Lemma 4.8.8 the sequences \( L^n c_1 P^n \) are in the algebra and by Lemma 4.8.9 

\[
L^n c P^n L^n W^0(\chi_\pm) P^n = L^n c W^0(\chi_\pm) P^n
\]

for \( c \in PC_2(\mathbb{R}) \), the verification is not difficult. Applying now the Stability Theorem to the sequence \( L^n P_\tau P R^n P_\tau + P^n Q_\tau Q^n \), the only difference to the Galerkin method is that instead of \( M \pm \) there appear the isomorphisms \( S_2 \pm \). We obtain then

\[
S^2_\pm(A_n) = \hat{Q} T^0 \left( c_1(\pm\infty) \frac{1 + \sigma_L}{2} + c_2(\infty) \frac{1 - \sigma_L}{2} \right) \hat{Q} + \hat{P} \\
S^{-2}_\pm(A_n) = \hat{P} T^0 \left( c_1(-\infty) \frac{1 + \sigma_L}{2} + c_2(-\infty) \frac{1 - \sigma_L}{2} \right) \hat{P} + \hat{Q}.
\]

\[\blacksquare\]

For \( d > 0 \), the results are even less complete than in the case \( d = 0 \). We will now give some information regarding approximation with piecewise linear splines (\( d = 1 \)). Consider the spline basis \( u^0(x) = \chi_{[0,1]}(x) \), \( u^1 = 2\sqrt{3}(x - \frac{1}{2})\chi_{[0,1]}(x) \). First we will derive some information regarding the function \( \sigma_P \) which by Theorem 4.1.2 verifies \( E_{n} P^n S_{\mathbb{R}} E_n = T^0(\sigma_P) \). Write

\[
\sigma_P = \begin{bmatrix} 
\sigma_{P00} & \sigma_{P01} \\
\sigma_{P10} & \sigma_{P11} 
\end{bmatrix}
\]

with \( \sigma_{P0k} \) representing the \( k \)th Fourier coefficient of \( \sigma_{P0} \). By section 2.2.3 in [17] we have that

\[
\sigma_{P00}(e^{2\pi iy}) = \frac{\sin^2(\pi y)}{\pi^2} \sum_j \frac{\operatorname{sgn}(-y-j)}{(y+j)^2}
\]

and it is possible to immediately derive that \( \sigma_{P0}^0 \) is real and \( \sigma_{P0}^0(1\pm) = \mp 1 \). Regarding the other entries of the matrix, we have by Theorem 4.1.2 that they are continuous at \( t = 1 \) (and at all other points \( t \in \mathbb{T} \)) and by [17, Theorem 2.15] that \( \sigma_{P0}^0(1) = 0 \). Regarding this last function note that its Fourier coefficients have to do with the image of \( u^0 \) by the Singular Integral Operator, that is, with the function

\[
(S_{\mathbb{R}} u^0)(x) = \frac{1}{\pi i} \log \left| \frac{x - 1}{x} \right|.
\]

It is possible to write them as

\[
\sigma_{P1k}^{10} = \int_{-\infty}^{+\infty} (S_{\mathbb{R}} u^0)(x) u^1_{k1}(x) \, dx = \frac{2\sqrt{3}}{\pi i} \int_k^{k+1} \log \left| \frac{x - 1}{x} \right| (x - k - \frac{1}{2}) \, dx.
\]

The Fourier coefficients of \( \sigma_{P}^{01} \) are related with the function

\[
(S_{\mathbb{R}} u^1)(x) = \frac{2\sqrt{3}}{\pi i} (1 + (x - \frac{1}{2}) \log \left| \frac{x - 1}{x} \right|)
\]
and can be written as
\[ \sigma_{PK} = \int_{-\infty}^{+\infty} (S_{IR}u^1)(x)u_{k1}^0(x) \, dx = \frac{2\sqrt{3}}{\pi} \int_k^{k+1} (1 + (x - \frac{1}{2}) \log \left| \frac{x - 1}{x} \right|) \, dx. \] (4.58)

As also
\[ \int_k^{k+1} 1 + (2x - k - 1) \log \left| \frac{x - 1}{x} \right| \, dx = 1 + \int_k^{k+1} 2x \log(x) - (k - 1) \log(x) \, dx + \int_k^{k+1} (k + 1) \log(x) - 2x \log(x) \, dx = 0, \]

and \( \sigma_{P0} = \frac{\sqrt{3}}{\pi} = -\sigma_{P0}^0, \sigma_{PK} = \sigma_{P-k}^0, \sigma_{P}^0 = \sigma_{P-k}^0 \) we obtain that (4.57) and (4.58) are conjugates, that is
\[ \sigma_{PK} = \overline{\sigma_{P-k}^0}, \]

and so
\[ \sigma_{P}^0(t) = \overline{\sigma_{P}^0(t)}, \quad t \in \mathbb{T}. \]

We are left with the function \( \sigma_{P}^{11} \), which is a continuous function, and its Fourier coefficients can be seen to have the property \( \sigma_{PK}^{11} = \sigma_{P-k}^{11}, \sigma_{P0}^{11} = 0 \) (because the integral is odd with respect to the point \( \frac{1}{2} \)) and this implies \( \sigma_{P}^{11}(1) = 0 \) and that \( \sigma_{P}^{11} \) is a real function. The determinant of the matrix function \( \sigma_{P} \) (the spectrum of the Laurent operator is given by the image of the determinant of \( \sigma_{P} \)) is then the real function \( \sigma_{P}^{00} \sigma_{P}^{11} - |\sigma_{P}^{10}|^2 \) which is continuous and takes the value 0 at the point 1.

We will now try to observe the behaviour of the corresponding function \( \sigma_{L} \) connected with the Collocation projection. We maintain the spline basis \( \{u^0, u^1\} \) and define a “collocation basis” as linear combinations of translated delta functionals. Let \( 0 < \epsilon < \frac{1}{2} \) and define the functionals
\[ \gamma^0 = \frac{1}{2} \delta_\epsilon + \frac{1}{2} \delta_{1-\epsilon}, \]
\[ \gamma^1 = -\frac{1}{2\sqrt{3}(1 - 2\epsilon)} \delta_\epsilon + \frac{1}{2\sqrt{3}(1 - 2\epsilon)} \delta_{1-\epsilon}. \]

These choice garanties good compatibility conditions with the spline basis, as we have
\[ \gamma^0(u^0) = 1, \quad \gamma^0(u^1) = 0, \]
\[ \gamma^1(u^0) = 0, \quad \gamma^1(u^1) = 1 \]

that is, the operator defined by \( E_{-n}L^nE_n : l_2 \to l_2^2 \) is the identity. If \( \gamma'_\epsilon = \delta_\epsilon \) (0 < \( \epsilon < 1 \), we have by [17, Theorem 2.17] (considering \( d = 0 \)) that the function associated with the Fourier coefficients
\[ \sigma_{Lk} = \gamma'_{+k}(S_{IR}u^0) = \delta_{+k}(S_{IR}u^0) \]
has the properties of being continuous for \( t \neq 1 \) and \( \sigma_L(1\pm) = \mp 1 \). Because \( \gamma^0 \) and \( \gamma^1 \) are just linear combinations of \( \gamma'_c \) and \( \gamma'_{1-c} \) it is possible to conclude immediately that \( \sigma_{L0}^{00}(1\pm) = \mp 1 \) and \( \sigma_{L0}^{10}(1\pm) = 0 \). Because \( \gamma^1 \) is “odd” with respect to the point \( \frac{1}{2} \) and \( S_{\mathbb{R}^I}u^1 \) is an integrable function, is monotonous away from zero and is even with respect to \( \frac{1}{2} \), it is also not difficult to see that \( \sigma_{L0}^{11}(1\pm) = 0 \).

We are left with \( \sigma_{L0}^{01} \). By observing the function \( S_{\mathbb{R}^I}u^1 \) it is obvious that the choice of the collocation point \( \epsilon \) influences (continuously) the value of \( \sigma_{L0}^{01}(1) \). If \( \epsilon \) is small enough, \( \sigma_{L0}^{01}(1) \) is negative. As \( \epsilon \) tends to \( \frac{1}{2} \), we can write

\[
\sigma_{L0}^{01}(1) = \frac{2\sqrt{3}}{\pi i} \sum_k (1 + k \log \left| \frac{k - \frac{1}{2}}{k + \frac{1}{2}} \right|) = \frac{2\sqrt{3}}{\pi i} \sum_k \sigma_k.
\]

We have \( \sigma_0 = 1 \) and \( \sigma_{-k} = \sigma_k \). Calculating the partial sum \( \sum_{k=1}^{n} \sigma_k \) it is obtained

\[
\sum_{k=1}^{n} \sigma_k = n + \log \left( \frac{1.3.5.\ldots 2n-1}{(2n+1)^n} \right) = n + \log \left( \frac{(2n-1)!}{(2n-1)! (2n+1)^n} \right)
\]

and by using the Stirling’s Formula \( n! \approx \left( \frac{n}{e} \right)^n \sqrt{2\pi n} \) we finally get

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \sigma_k = \lim_{n \to \infty} \left( n \log \left( \frac{2n+1}{2n} \right) - 1 + \log \left( \sqrt{\frac{2n+1}{2n}} \right) \right) = -\frac{1}{2} + \frac{1}{2} \log(2) > -\frac{1}{2}
\]

which means that \( \sigma_{L0}^{01}(1) > 0 \) for \( \epsilon \) near \( \frac{1}{2} \). It is then possible to conclude that there is at least a choice of the collocation point \( \epsilon \) such that \( \sigma_{L0}^{01}(1) = 0 \), which implies \( \sigma_L - \sigma_P \) is a continuous matrix function on \( \mathbb{T} \) that takes the value 0 at the point 1 and so the collocation sequence is in \( \mathcal{A} \).

### 4.9 Final remarks

As in the finite section chapter, for the sake of simplicity, only scalar operators were considered as generators of the algebra \( \mathcal{A} \). But if we consider \((W^0(a))\) and \((cI)\) with \(a\) or \(c\) in \([PC(\mathbb{R})]^{n \times n}\), the proofs remain the same. This covers the operators related with systems of singular integral equations or systems of Wiener-Hopf operators. Obviously the operators that result from the homomorphisms have then matrix coefficients (and the Laurent operators turn into matrices of Laurent operators), and can be difficult in the general case to find invertibility conditions for these operators. For a non scalar version of the Two Projections Theorem see [14].

Also it is possible to consider Wiener-Hopf plus Hankel type of operators, as was done in the finite section case. Just include \((J)\) in \( \mathcal{A} \), with \((Ju)(t) = u(-t) \). Then it is necessary to make the localization with only even continuous functions and use the flip elimination method described in Lemma 2.4.1.
Chapter 5

Around non-uniform meshes

5.1 Introduction

For all the nice results obtained in the last two chapters, it remains to emphasize the fact that, in practice, uniform partitions are only seldom used. This derives from space limitations in terms of computer memory and speed. For example, if one has a Wiener-Hopf equation, the behaviour of the solution near the critical point 0 is usually more important than that near the point 60. But using an uniform mesh with the required finess near the point 0 would mean that the same finess would have to be used for all other parts of the domain, leading to a huge linear system. So the idea is to use a non-uniform mesh, with smaller intervals between the mesh points only on the more interesting parts of the domain.

The application of Banach algebra techniques for non-uniform meshes has some particular difficulties. Due to size and time limitations, it was not possible in this work to obtain the same type of results for non-uniform meshes as was done with uniform meshes. The author opted for introducing some types of mesh and give some explanation of the difficulties and research paths open.

There are two basic ways to attack the non-uniform meshes. One is to consider the mesh “as it is” and try to follow the algebraization, essentialization, localization and identification procedures. The other is to find a transformation that “uniformizes” the mesh, and then look at the resulting operators and study the modified operators (and spline spaces) related to the (now) uniform mesh. The first was the one followed on this chapter, but the second seems promising and will be the object of further research by the author. To make things more clear regarding the second way let’s introduce the meshes
\[ \Delta_{q,1}^n = \left\{ x_j^{(n)} = \text{sgn}(j)q \log \left( \frac{n}{n - |j|} \right), -n < j < n \right\}, \quad q > 0 \] (5.1)

\[ \Delta_{q,2}^n = \left\{ x_j^{(n)} = \text{sgn}(j) \left( \frac{|j|}{n} \right)^q, -\infty < j < +\infty \right\}, \quad q \geq 1 \] (5.2)

\[ \Delta_{q,3}^n = \left\{ x_j^{(n)} = iq \frac{1 - e^{2\pi i j}}{1 + e^{2\pi i j}}, -\frac{n}{2} < j < \frac{n}{2} \right\}, \quad q > 0 \] (5.3)

with associated projections \( P_1^n, P_2^n \) and \( P_3^n \) to the piecewise constant spline space. Introducing now the unitary transformations

\[ T_{q,1} : L^2(\mathbb{R}) \to L^2([-1, 1]), \quad (T_{q,1}u)(x) = \text{sgn}(x) \sqrt{\frac{q}{|x|}} u(-\log(|x|)) \] (5.4)

\[ T_{q,2} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (T_{q,2}u)(x) = \text{sgn}(x) \sqrt{|x|^{\frac{q-1}{2}}} u(\text{sgn}(x)|x|^q) \] (5.5)

\[ T_{q,3} : L^2(\mathbb{R}) \to L^2(\mathbb{T}), \quad (T_{q,3}u)(t) = \frac{i\sqrt{2q}}{1 + t} u(iq \frac{1 - t}{1 + t}) \] (5.6)

and considering the singular integral operator \( S_\mathbb{R} \) it is not difficult to see that for \( k = 1, \ldots, 3 \) the method

\[ P_1^n S_\mathbb{R} P_1^n u = P_1^n v \] (5.7)

is equivalent to the method

\[ T_{q,k} P_1^n T_{q,k}^{-1} T_{q,k} S_\mathbb{R} T_{q,k}^{-1} T_{q,k} P_1^n T_{q,k}^{-1} \] (5.8)

which uses a uniform mesh. The operator \( T_{q,k} S_\mathbb{R} T_{q,k}^{-1} \) is a Mellin type operator for \( k = 1 \) or \( k = 2 \) and for \( k = 3 \) is the singular integral operator on the unit circle. The main problem is that the projections \( T_{q,k} P_1^n T_{q,k}^{-1} \) are no longer into the piecewise constant spline space, but into a spline space that is not invariant by translation (see figure 5.1). But if it is possible to find convergence criteria for this type of methods and operators then the convergence criteria is valid for the case that interest us.

\textbf{Remark 5.1.1.} Note that the converse is true. For example, the fact that in the previous chapter we found out convergence criteria for the singular integral operator with uniform meshes, means that these criteria are also true for certain Mellin operators with non-uniform meshes and certain spline spaces which are not invariant by translation (see also [17, Section 6.3.1]).

We will now in the rest of this chapter illustrate the first methodological option with a mesh of the type of the mesh \( \Delta_{q,1}^n \), and the Galerkin projection.
5.2 Multiindiced sequences

In the study of approximation methods with graded meshes, one of the problems that appear immediately is that there is no guarantee that the length of the maximal interval between two mesh points converges to zero as $n \to \infty$. Because this was an argument used in several places when studying uniform meshes (see for instance the proof of Lemma 4.5.1 where it was used the fact that the space between two mesh points was always equal to $1/n$) it is necessary to introduce a new concept, the theory of multiindiced operator and approximation sequences. For a short review of these ideas, the reader is directed to [17, Section 4.2.2]. Consider the pair $\{\mathbb{Z}_+^2, \preceq\}$ with the partial ordering defined as $t_1 \preceq t_2$, $t_1 = (n_1, i_1)$, $t_2 = (n_2, i_2)$ if and only if $n_1 \leq n_2$ and $i_1 \leq i_2$. Now choose $\mathcal{T} \subset \mathbb{Z}_+^2$ such that $\mathcal{T}$ is a directed set, i.e. given any two elements $t_1$, $t_2$ of $\mathcal{T}$, there exists always a different third element $t_3$, such that $t_1 \preceq t_3$ and $t_2 \preceq t_3$. Also $\mathcal{T}$ must fulfill the following two conditions: For any monotonically increasing sequence $(t_k)_{k \in \mathbb{N}} \subset \mathcal{T}$ the quotient $n_k/i_k$ tends to infinity, and the set $\{i : (n, i) \in \mathcal{T}\}$ is unbounded. For any $\mathcal{T}' \subset \mathcal{T}$ we say that a sequence of operators $(A_t)$ converges strongly in $\mathcal{T}'$ to the operator $A$, and represent this by

$$A_t \to A_{\mathcal{T}'} \text{ or } s\lim_{t \in \mathcal{T}'} A_t = A \quad (5.9)$$

if for any monotonically increasing sequence $(t_k)_{k \in \mathbb{N}} \subset \mathcal{T}'$ the sequence $A_{t_k}$ tends strongly to $A$ in the usual sense of the word. We will use the norm convergence related

\footnote{understood here always as strongly monotonically increasing}
The sequence \((A_t)\) will be said to converge in the norm (or uniformly) to an operator \(A\) if and only if for any \(\epsilon > 0\) there exists a \(t_0 \in \mathbb{T}\) such that for all \(t \geq t_0\) the norm \(\|A - A_t\|\) is less than \(\epsilon\). Note that using these definitions, results similar to lemmas 2.1.1 or 2.1.9 are true, as well as the Banach-Steinhaus Theorem. Consider now the mesh sequence \((\Delta_t)_{t \in \mathbb{T}}\), with

\[
\Delta^t = \left\{ x_j^{(n)} = \text{sgn}(j) \log \left( \frac{n}{n - |j|} \right), -n + i < j \leq n - i \right\}
\]  

and define the corresponding intervals between mesh points

\[
J_j = \left[ \log \left( \frac{n}{n - j + 1} \right), \log \left( \frac{n}{n - j} \right) \right]
\]

for \(j\) positive and

\[
J_j = \left[ -\log \left( \frac{n}{n + j - 1} \right), -\log \left( \frac{n}{n + j} \right) \right]
\]

when \(j\) is zero or negative.

Our objective is to study the Galerkin projection \(P^t\) associated with this mesh and the piecewise polynomial spline space

\[
S^t = \{ u \in L^2(\mathbb{R}) : u|_{J_j} \in \mathbb{P}^d(J_j), u|_{[\log(\frac{n}{n - 1}), \log(\frac{n}{n})]} = 0 \},
\]

where \(\mathbb{P}^d\) represents the set of polynomials of degree less or equal to \(d\). Note that we considered \(q = 1\), for our purposes without loss of generality because if \(q \neq 1\) we need just to multiply all intervening operators and functions by “scaling” operators \(Z_q\) and \(Z_q^{-1}\) (defined in (2.21)) to reduce the mesh to \(q = 1\).

There are several results in this chapter dealing with strong convergence and norms, like the one below, that are already known in a general sense. But as it is difficult to find references for the particular conditions that are being dealt here with, the author opted to include proofs, which are adapted from the ones presented in, for example, [28, Chapter 5].

**Proposition 5.2.1.** The operators \(P^t\) are uniformly bounded in \(L^2(\mathbb{R})\) and we have \(s\)-\(\lim_{t \in \mathbb{T}} P^t = I\).

**Proof:** Let \(P : L^2[0, 1] \to \mathcal{P}_d\) represent the orthogonal projection in the space \(L^2[0, 1]\) onto \(\mathcal{P}_d\). It is not difficult to see that\(^2\)

\[
\|Pv\|_{[0, 1]} \leq \|v\|_{[0, 1]}.
\]

If we define the function \(v(x) = u(x_{j-1} + h_j x)\) with \(h_j = x_j - x_{j-1}\), we can then define \(P^t\) as

\[
(P^t u)(x) = \begin{cases} (Pv)(h_j^{-1}(x - x_{j-1})) & x \in J_j, -n + i < j \leq n - i, \\ 0 & x < x_{-n+i} \text{ or } x > x_{n-i} \end{cases}
\]

\(^2\)In order to have a less heavy notation, we will omit the dependence on \(n\) of the mesh points, when it is not important.
and this implies that for any \( j \) between \(-n+i+1 \) and \( n-i \), \( \|P^t u\|_{J_j} \leq \|u\|_{J_j} \).

To prove the strong convergence, suppose first that the function \( u \) is a continuously differentiable real function of compact support (with \( u(x) = 0 \) for \( |x| > \tau_0 \)). Then we have, for some \( \tau_1 \geq \tau_0 \)

\[
\|(I - P^t)u\|^2 = \int_{-\infty}^{+\infty} |u(x) - (P^t u)(x)|^2 \, dx = \int_{-\tau_1}^{\tau_1} |u(x) - (P^t u)(x)|^2 \, dx.
\]

Due to the fact that \( P^t \) is an orthogonal projection, we can majorate this norm for general \( d \), by the norm for \( d = 0 \) (the projection into the space of piecewise constant functions, constant in each \( J_j \)) and obtain

\[
\int_{-\tau_1}^{\tau_1} |u(x) - (P^t u)(x)|^2 \, dx \leq \sum_{j=1}^{k} \int_{x_{j-1}}^{x_j} |u(x) - u(x_0^j)|^2 \, dx
\]

for some \( k \) depending on \( \tau_0 \) and \( n-i \), and \( x_0^j \) between \( x_{j-1} \) and \( x_j \) due to the Mean Value Theorem. We have then using the Lipschitz continuity of \( u \) that

\[
\sum_{j=-k+1}^{k} \int_{x_{j-1}}^{x_j} |u(x) - u(x_0^j)|^2 \, dx \leq c \int_{x_{j-1}}^{x_j} |x - x_0^j|^2 \, dx \leq c \left( \max_{-k<j\leq k} h_j^2 \right) \tau_1
\]

and this last expression tends to zero as \( n \) goes to infinity. As any complex valued function can be written as a sum \( u = u_1 + i u_2 \), with \( u_1, u_2 \) real valued, the result is true for continuous any differentiable function with compact support.

If \( u \in L_2(\mathbb{R}) \), it can be uniformly approximated by a sequence \((u_k)\) of differentiable functions with compact support. Now, given \( \epsilon > 0 \) there is a \( t_0 \) such that for any \( t \geq t_0 \) we can find an \( m \) so that

\[
\|u - P^t u\| \leq \|u - u_m\| + \|u_m - P^t u_m\| + \|P^t(u_m - u)\|
\]

and each of the three norms on the righthand side is less then \( \epsilon/3 \). This means that \( s\text{-lim}_{t \in \mathbb{T}} \|u - P^t u\| = 0 \).}

We will define now what is meant by convergence (or applicability) of an approximation method when using multiindiced sequences. The approximation method

\[
A_t u_t = v \tag{5.11}
\]

is said to apply to the operator \( A \) if there exists a \( t_0 \), such that for all \( t \geq t_0 \) and \( v \in L_2(\mathbb{R}) \) the equation \((5.11)\) has a unique solution, and this solution converges in the \( L_2 \) norm to a solution of the equation \( Au = v \) for each monotonically increasing sequence \((t) = (t_k)_{k \in \mathbb{N}} \subset \mathbb{T} \).
5.3 Algebraization

We define now the algebra $E$ of all sequences such that $\sup_{t \in T} \|A_t\|_{L(L^2)} < \infty$ and the ideal $G \subset E$ of the sequences which tend uniformly to zero (i.e. for any $\epsilon > 0$, there is always a $t_0 \in T$ such that for all $t \geq t_0$ the norm of the operators are less than $\epsilon$). Moreover we say that a sequence $(A_t) \in E$ is stable if there exists a $t_0$, such that for all $t \geq t_0$ $A_t$ is invertible, and $\sup_{t \geq t_0} \|A_t^{-1}\|_{L(L^2)} < \infty$.

As in cases already treated, it is possible to prove the equivalences stated in the next theorem (see also [17, Section 4.2.3]).

**Theorem 5.3.1.** The following propositions are equivalent for $A_t \in E$ such that $A_t \rightarrow A$ in $L(L^2(\mathbb{R}))$:

(a) The approximation method (5.11) applies to $A$;
(b) $(A_t)$ is stable and $A$ is invertible;
(c) $A$ is invertible and the coset $(A_t) + G$ is invertible in the quotient algebra $E/G$.

5.4 Essentialization

Assign to each $t = (n, i) \in T$ the number $\tau = \tau(t) = \log(n)$ (note that $\tau_{n-i}^{(n)} = \log(n)$) and define the operators $P_t := P_{\tau}, Q_t := Q_{\tau}, R_t := R_{\tau}, V_t := V_{\tau}$ and $V_{-t} := V_{-\tau}$. Let now $T' \subset T$ be such that for any monotonically increasing sequence $(t_k)_{k \in \mathbb{N}} \subset T'$ we have $i_k \rightarrow \infty$. An example of such a set is

$$\left\{ (n, i) \in T : n^\epsilon < i < n^{1-\epsilon}, \ 0 < \epsilon < \frac{1}{2} \right\}.$$  

It must be remarked that the introduction of the set $T'$ is just a technical necessity in order to define strong limits. The sequences are still related with the set $T$.

Consider now the following strong limits in $T'$:

$$A_t \rightarrow A \text{ and } A_t^* \rightarrow A^*;$$

$$R_tA_tA_t \rightarrow A_{11} \text{ and } (R_tA_tA_t)^* \rightarrow A_{11}^*;$$

$$R_tA_tV_t \rightarrow A_{12} \text{ and } (R_tA_tV_t)^* \rightarrow A_{12}^*;$$

$$V_{-t}A_tA_t \rightarrow A_{21} \text{ and } (V_{-t}A_tA_t)^* \rightarrow A_{21}^*;$$

$$V_{-t}A_tV_t \rightarrow A_{22} \text{ and } (V_{-t}A_tV_t)^* \rightarrow A_{22}^*.$$  \hspace{1cm} (5.12)

Let $F \subset E$ be the $C^*$-subalgebra generated by all constant sequences $A_t = A$ for which the strong limits above exist, the finite section projection sequence $(P_t)$, the sets $J_0$.
and $\mathcal{J}_1$ defined by formula 3.2, with $\tau$ substituted by $t$ and the Galerkin sequence $(P^t)$. It is not difficult to verify that the strong limits exist also for $(P^t)$.

But in order to prove that $\mathcal{J}_0$ and $\mathcal{J}_1$ are really closed two sided ideals also in this case, the following auxiliary result is necessary:

**Lemma 5.4.1.** Let $K \in \mathcal{K}$. Then $\lim_{i \to \infty} \sup_{n \in \mathbb{N}} \| (I - P^t)R_tK \|_{\mathcal{L}(L^2)} = 0$.

**Proof:** Fix $i$, put $\tau = \log(\frac{2}{t})$ and put $a(i) := \sup_{n \in \mathbb{N}} \| (I - P^t)R_tu \|_{L^2}$. We will begin by proving that $a(i) \to 0$ as $i \to \infty$. Suppose first $u$ continuously differentiable with support $\subset [-\tau_1, \tau_1]$. Using a similar process to the one used to prove Proposition 5.2.1, it can be seen for $\tau > \tau_1$ that $\| (I - P^t)R_tu \|_{L^2}^2 \leq c(h_j^{(n)})^2$ for $j$ such that $J_j \cap ([\tau, 1 - \tau_1) \cup [\tau - 1, \tau) \neq \emptyset$ and $\| (I - P^t)R_tu \|_{L^2}^2 = 0$ in the other cases. Then there exists a $l$ ($\leq n - i$) such that

$$\| (I - P^t)R_tu \|_{L^2}^2 \leq c \sum_{k=-l+1}^l (h_k^{(n)})^2 \leq c \left( \max_{-n+1 < j \leq n-i} h_j^{(n)} \right)^2 \sum_{k=-l+1}^l h_k^{(n)} \leq \left( \max_{-n+1 < j \leq n-i} h_j^{(n)} \right)^2 (2\tau_1 + 2 \max_{-n+1 < j \leq n-i} h_j^{(n)})^2.$$ 

As for each mesh sequence of this type there exists a positive constant $c_1$ independent of $j$ and $n$ such that

$$h_j^{(n)} \leq c_1 \frac{1}{n - |j|}$$

(see [28, Remark 5.6]), we get

$$\max_{-n+1 < j \leq n-i} h_j^{(n)} \leq c_1 \frac{1}{n - (n - i)} = c_1 \frac{1}{i}$$

which is independent of $n$ and goes to zero as $i \to \infty$.

Now we can use also the arguments in the proof of Proposition 5.2.1 to extend the result for any function in $L^2(\mathbb{R}^+)$ due to the fact that $(I - P^t)R_t$ is uniformly bounded, and we proved that $a(i) \to 0$ as $i \to \infty$. The result now follows from a similar argument to the one used to prove that if a sequence of operators, $(A_k)$, converges strongly to zero, then $\|A_kK\| \to 0$ for $K \in \mathcal{K}$ (see, for instance [28, 1.1(h)]).

**Proposition 5.4.2.** The sets $\mathcal{J}_0$ and $\mathcal{J}_1$ are closed two sided ideals of $\mathcal{F}$.

**Proof:** The proof is similar to the finite section case (propositions 3.3.1 and 3.3.2) for the constant sequences and $P_t$, taking into account that for these sequences, the strong limits exist not only in $\mathbb{T}'$ but also in $\mathbb{T}$. For the Galerkin projection sequence $(P^t)$ it is necessary to use Lemma 5.4.1. For example,

$$P^tR_tK_1R_t = R_tK_1R_t - (I - P^t)R_tK_1R_t$$
and \((I - P^t) R_t K_1 R_t\) is in \(\mathcal{G}\) because of Lemma 5.4.1. The closedness proof is also similar to the finite section case.

Now it is possible to define the homomorphisms \(O^0\) and \(O^1\) (as strong limits in \(\mathcal{T}'\)), and apply the Lifting Theorem as in the finite section case (see formulas (3.3) and (3.4)).

So let now the algebra \(\mathcal{A} \subset \mathcal{F}\) be generated by the finite section projection sequence \((P_t)\), the constant sequences \((cI)\) with \(c \in PC_{(0)}(\mathbb{R})\), the constant sequences \((W^0(a))\) with \(a \in PC(\mathbb{R})\) and the Galerkin projection sequence \((P^t)\).

5.5 First localization and identification

We will start by two commutator results.

Lemma 5.5.1. Let \(f \in C(\mathbb{R})\) such that \(f\) is uniformly continuous. Then for any monotonically increasing sequence in \(\mathcal{T}'\), \(\|fP^t - P^t f\|_{L^2(\mathbb{R})} \to 0\).

Proof: Let \(\{u^l_j, 0 \leq l \leq d, -n + i < j \leq n - i\}\) be an orthonomal basis for the spline space \(S^t\) (and which depend on \(t\)) such that the functions \(u^l_j\) are real and \(\text{supp}(u^l_j) \subset J_j\), let \(f\) be real and let \(v\) be a positive function. Then, using the Mean Value Theorem and writing \(u^l_j = u^l_{j+} - u^l_{j-}\), with \(u^l_{j\pm} \geq 0\), we obtain

\[
\|(fP^t - P^t f)v\|_{L^2(\mathbb{R})}^2 = \sum_{j=-n+i+1}^{n-i} \left( \int_{J_j} \left| f(s) - f(s_{j+}) \right| \left( \sum_{l=0}^{d} \int_{J_j} v(t)u^l_{j+}(t) dt \right) \right)^2 ds = \\
= \sum_{j=-n+i+1}^{n-i} \left( \int_{J_j} \left| f(s) - f(s_{j-}) \right| \left( \sum_{l=0}^{d} \int_{J_j} v(t)u^l_{j-}(t) dt \right) \right)^2 ds \leq (5.13)
\]

with \(s_{j\pm} \in J_j\). The above is less or equal than

\[
\sum_{j=-n+i+1}^{n-i} \left( \int_{J_j} \left| f(s) - f(s_{j+}) \right| \left( \sum_{l=0}^{d} \int_{J_j} v(t)u^l_{j+}(t) dt \right) \right)^2 ds + \left| f(s) - f(s_{j-}) \right| \left( \sum_{l=0}^{d} \int_{J_j} v(t)u^l_{j-}(t) dt \right)^2 ds \leq (5.14)
\]
Lemma 5.5.2. Let

\[ \omega(f, h^{(n)}_{j}) \sum_{i} \int_{J_{j}} \left| \omega(f, h^{(n)}_{j}) \sum_{i} \left( \int_{J_{j}} |v(t)|^2 dt \right) \left( \int_{J_{j}} |u^{+}_{j}(t)|^2 dt \right) \left( \int_{J_{j}} |u^{-}_{j}(t)|^2 dt \right) \left| u^{j}_{j}(s) \right| ds. \]  

with

\[ \omega(f, e) = \sup \{ |f(s) - f(t)| : |s - t| < e \}. \]  

Using now the fact that \( \int_{J_{j}} |u^{+}_{j}(t)|^2 dt \leq 1 \) and the property \( (\sum_{0}^{d} |a_{i}|)^2 \leq (d+1) (\sum_{0}^{d} |a_{i}|)^2 \) we obtain that (5.15) is less or equal than

\[ 4\omega(f, \max_{-n+i < j \leq n-i} (h^{(n)}_{j}))^2 \sum_{j=-n+i+1}^{n-i} \left( \int_{J_{j}} |v(t)|^2 dt \right) \left( \int_{J_{j}} \sum_{l=0}^{d} \left| u^{l}_{j}(s) \right| ds \right)^2 \leq \]

\[ 4\omega(f, \max_{-n+i < j \leq n-i} (h^{(n)}_{j}))^2 (d+1)^2 \sum_{j=-n+i+1}^{n-i} \left( \int_{J_{j}} |v(t)|^2 dt \right) \left( \int_{J_{j}} \sum_{l=0}^{d} \left| u^{l}_{j}(s) \right|^2 ds \right). \]

As \( \int_{J_{j}} \left| u^{l}_{j}(s) \right|^2 ds = 1 \) the expression above simplifies to

\[ 4\omega(f, \max_{-n+i < j \leq n-i} (h^{(n)}_{j}))^2 (d+1)^3 \sum_{j=-n+i+1}^{n-i} \left( \int_{J_{j}} |v(t)|^2 dt \right) = \]

\[ = 4(d+1)^3 \omega(f, \max_{-n+i < j \leq n-i} (h^{(n)}_{j}))^2 \| v \|_{L^2(\mathbb{R})}^2, \]

with \( \omega(f, \max_{-n+i < j \leq n-i} (h^{(n)}_{j})) \) tending to zero for each monotonically increasing sequence \( (t_{k}) \in \mathbb{T}' \). To end the proof we remark that any function in \( L^2(\mathbb{R}) \) can be written as a difference of two positive functions and that any continuous function \( f \) as \( f_{1} + i f_{2} \), with \( f_{1,2} \) real.

\[ \Box \]

**Lemma 5.5.2.** Let \( f \in C(\mathbb{R}) \). Then \( \| f P^{t} - P^{t} f \|_{L^2(\mathbb{R})} \) tends uniformly to zero (in relation to the index set \( \mathbb{T} \)).

**Proof:** Let \( C'(\mathbb{R}) \) be the subalgebra of \( C(\mathbb{R}) \) generated by the continuous functions \( f \) such that \( f(s) = f(\infty) \) for \( |s| \) greater than some constant \( d \) (which can vary from function to function). If \( f \) is one of the generators of \( C'(\mathbb{R}) \), then \( f(s) \) is constant for \( |s| > d \) and we can ignore all intervals \( J_{j} \) totally contained in the exterior of the interval \([ -d, d] \). For the interior, it is possible to use a proof similar to the above lemma, because it has to do with the greater interval between two mesh points. This interval depends again only on \( n \) and so by increasing \( n \) one can have \( \| f P^{t} - P^{t} f \|_{L^2(\mathbb{R})} \) as
small as desired. The result is true then for any element of \( C'(\mathbb{R}) \). But by the Stone-Weierstrass Theorem the closure of \( C'(\mathbb{R}) \) is \( C(\mathbb{R}) \) and the result follows.

Having proved that the subalgebra generated by the cosets \( fI + \mathcal{J} \) with \( f \) continuous in \( \mathbb{R} \) belongs to the center of \( \mathcal{A}^{\mathcal{J}} \), the localization procedure now follows the same steps as in the previous chapters. The following result is proved in a similar way to Lemma 4.4.3.

**Lemma 5.5.3.** The algebra \( \mathcal{A}_0^{\mathcal{J}} \) is generated by the identity and the cosets \( \Phi_0^J(P^t), \Phi_0^J(\chi_+) \) and \( \Phi_\varepsilon^J(W^0(\chi_+)) \). For \( x \in \mathbb{R} \setminus \{0\} \), the algebra \( \mathcal{A}_x^{\mathcal{J}} \) is generated by the identity and the cosets \( \Phi_0^J(P^t) \), and \( \Phi_\varepsilon^J(W^0(\chi_+)) \). All cosets are projections.

For the local algebras above, contrary to what was done up to now, it was not possible to find an isomorphism (a local equivalent representation) that would identify these algebras. The local algebras corresponding to \( x \neq 0 \) are generated by only two idempotents, and so it is possible to use the Two-Projections Theorem to identify them. But the algebra \( \mathcal{A}_0^{\mathcal{J}} \) is generated by three idempotents. In general algebras generated by three idempotents (even if two of them commute, as in this case) are very difficult to study (see for instance [2] were algebras generated by idempotents were studied). Nevertheless it is possible here to define an homomorphism from the local algebra to an operator algebra. Introduce the following transformation \( O_{xt} \) such that to the sequence \((A_t)\) associates

\[
O_{xt}(A_t) = Z_{ne^{-|x|}} \tilde{V}_{-x} A_t \tilde{V}_x Z_{ne^{-|x|}}^{-1}
\]

where \( x_n = \text{sgn}(j_n(x)) \log\left(\frac{n}{|j_n(x)|}\right) \) is the mesh point in \( \Delta^t \) nearer to the point \( x \), which means that \( j_n(x) \) is the integer nearer to \( \text{sgn}(x)n(1 - e^{|x|}) \). When \( \log(n) > |x| \) we will have that \( |x - x_n| < 2e^{|x|}/n \). We introduce now the homomorphism

\[
O_x(A_n) = \text{s-lim}_{t \to \mathcal{T}} O_{xt}(A_t)
\]

for the sequences for which it exists.

**Lemma 5.5.4.** The strong limits \( O_x(A_t) \) exist for all \((A_t) \in \mathcal{A}\). In particular,

(a) \( O_x(P_t) = I, O_x(P_t^t) = P^1 \);

(b) \( O_x(cI) = c(x^-)\chi_- + c(x^+)\chi_+ \), \( c \in PC_{(0)}(\mathbb{R}) \);

(c) \( O_x(W^0(a)) = a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+) \);

(d) if \((j_t) \in \mathcal{J}_0 \) or \((j_t) \in \mathcal{J}_1 \), then \( O_x(j_t) = 0 \).

**Proof:** For the second part of (a) one can think of \( \tilde{V}_{-x} P^t \tilde{V}_x \) as a projection associated with the mesh \( x_j^{(n)} = \text{sgn}(j) \log\left(\frac{n}{|j|}\right) - \text{sgn}(j_n(x)) \log\left(\frac{n}{|j_n(x)|}\right) \). Then \( O_{xt}(P^t) \) is the projection with the mesh

\[
\frac{n}{e^{|x|}} \left( \text{sgn}(j) \log\left(\frac{n}{n-j}\right) - \text{sgn}(j_n(x)) \log\left(\frac{n}{n-j_n(x)}\right) \right)
\]
and if we calculate the limit of the above expression as \( n \) increases, the limit is \( j - j_n(n) \). The assertions (b), (c), (d) and the first part of (a) are proved as in the uniform mesh case.

If \( x \neq 0 \), as already mentioned, the local algebras \( A_x^J \) are generated by two idempotents, \( \Phi_x^J(P^t) \) and \( \Phi_x^J(W^0(\chi_+)) \). To use the Two Projections Theorem, as we have already seen, one needs to know the local spectrum

\[
\sigma_{A_x^J}(\Phi_x^J(P^tW^0(\chi_+)P^t)).
\]

Due to the fact that \( (P^tW^0(\chi_+)P^t) \) is a positive definite sequence with norm less or equal than 1, and the existence of the homomorphism \( O_x \), we can write

\[
\sigma_B(O_x(P^tW^0(\chi_+)P^t)) \subseteq \sigma_{A_x^J}(\Phi_x^J(P^tW^0(\chi_+)P^t)) \subseteq [0, 1]. \tag{5.19}
\]

Now the result

\[
\sigma_{A_x^J}(\Phi_x^J(P^tW^0(\chi_+)P^t)) = [0, 1]. \tag{5.20}
\]

comes as in the finite section chapter (see the paragraph before Proposition 3.4.11). Applying the Two Projections Theorem ([17, Theorem 1.10]) we obtain an isomorphism \( N \) to a space of matrix functions defined on \([0, 1]\),

\[
N(\Phi_x^J(I)) = e^t : s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \tag{5.21}
\]

\[
N(\Phi_x^J(P^t)) = p^t_1 : s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \tag{5.22}
\]

\[
N(\Phi_x^J(W^0(\chi_+))) = p^t_2 : s \mapsto \begin{bmatrix} s & \sqrt{s(1-s)} \\ \sqrt{s(1-s)} & 1-s \end{bmatrix}. \tag{5.23}
\]

If \( x = 0 \), in the general case, we have only a set of necessary conditions for invertibility in the local algebra. But if we restrict the operators involved (considering continuity of the symbols at 0 for the multiplication or the convolution operators, then we fall again in a algebra generated by two idempotents and the solution above (for \( x \neq 0 \)) can be applied to obtain a set of necessary and sufficient conditions. Other possibility to overcome this difficulty at \( x = 0 \) is to consider only Wiener-Hopf operators, avoiding the necessity of this localization altogether. This will be done at the end of the chapter.

## 5.6 Second localization and identification

We start again with a few auxiliary results.

**Lemma 5.6.1.** If \( g \in C(\mathbb{R}) \) and \( g(\infty) = 0 \), the sequences \( (P_tQ^tW^0(g)) \) and \( (W^0(g)P_tQ^t) \) are in \( G \), with \( Q^t = I - P^t \).
Proof: We can approximate $W^0(g)$ in the operator norm by operators $W^0(g_m)$ such that $k_m = F^{-1}g_m$ are functions belonging together with their derivatives to $L^1(\mathbb{R})$. We then get using the same proof as in \cite{28} Lemma 5.25 that there exists a constant $d$ such that
\[ \|Q^tW^0(g_m)u\|_{L^2} \leq dh_j\|DW^0(g_m)u\|_{L^2}, \]
the $D$ representing the operator of derivation and $h_j$ the length of the interval $J_j$. So we obtain
\[ \|P_tQ^tW^0(g_m)u\|_{L^2} \leq d\max(h_j)\|DW^0(g_m)u\|_{L^2} \leq d'\max(h_j)\|u\|_{L^2}, \]
for some constants $d, d'$, the last inequality due to the continuity of $W^0(g_m)$ from $L^2(\mathbb{R})$ to $L^{2,1}(\mathbb{R})$. As $\max(h_j), -n + i \leq j \leq n - i$ can be made as small as desired by increasing $i$ the result that $(P_tQ^tW^0(g)) \in \mathcal{G}$ follows. To prove the result for the sequences $(W^0(g)P_tQ^t)$ it is sufficient to use a duality argument, since $Q^t\ast = Q^t$. ■

Using the last lemma we can now prove that the cosets $\Phi^J_\infty(W^0(g))$ with $g$ continuous are in the center of our algebra.

Proposition 5.6.2. If $g \in C(\overline{\mathbb{R}})$ then the coset $\Phi^J_\infty(W^0(g))$ commutes with all elements of $\mathcal{A}^\mathcal{J}_\infty$.

Proof: Due to the results of Proposition 3.4.6 in the finite section case, we only need to verify the result for the coset $\Phi^J_\infty(P_t)$. It is possible to write
\[ P_tW^0(gI) - W^0(gI)P_t = P_tW^0((g - g(\infty))I)Q^t - Q^tW^0((g - g(\infty))I)P_t = P_tW^0((g - g(\infty))I)P_tQ^t - P_tQ^tW^0((g - g(\infty))I)P_t + j_t \] (5.24)
with $(j_t) \in \mathcal{J}$ because $Q_tQ^t = Q_t$, and as $g - g(\infty)$ is a continuous function that takes the value 0 at the point infinity we can apply Lemma 5.6.1 and see that the sequence $(P_tW^0((g - g(\infty))I)Q^t - Q^tW^0((g - g(\infty))I)P_t) \in \mathcal{J}$. This last result means we can localize $\mathcal{A}^\mathcal{J}_\infty$ over the maximal ideal space of the subalgebra $\mathcal{C}$ generated by the cosets $\Phi^J_\infty(W^0(g))$ with $g \in C(\overline{\mathbb{R}})$. This maximal ideal space is formed by the cosets $\Phi^J_\infty(W^0(g_x))$ with $g_x(x) = 0$ and $x \in \overline{\mathbb{R}}$, and is isomorphic to $\mathbb{R}$. That these maximal ideals are indeed not trivial, can be verified by the homomorphisms $S_y$, defined below. ■

In order to apply the Local Principle of Allan, let $\mathcal{I}_{\infty,y}, y \in \overline{\mathbb{R}}$, be the smallest closed two sided ideal of $\mathcal{A}^\mathcal{J}_\infty$ which contains the ideal $y$ of $\mathcal{C}$. We call $\Phi^\mathcal{J}_{\infty,y}$ the homomorphism which is the composition of the canonical homomorphism from $\mathcal{A}^\mathcal{J}_\infty$ to $\mathcal{A}^\mathcal{J}_{\infty,y} := \mathcal{A}^\mathcal{J}_\infty/\mathcal{I}_{\infty,y}$, with $\Phi^\mathcal{J}_\infty$.

To identify the local algebras we introduce the application $S_{yt}$ that to a sequence $(A_t) \in \mathcal{A}$ associates
\[ S_{yt}(A_t) = Z_{\tau}^{-1}U_{-y}A_tU_yZ_{\tau}, \quad \tau = \log\left(\frac{n}{t}\right) \] (5.25)
5.7. THE LOCAL ALGEBRA $\mathcal{A}_{J,\infty,\infty}$

For the sequences for which it exists, we define the strong limit

$$S_y(A_t) = s\text{-lim}_{t \to \infty} S_{yt}(A_t)$$  \hspace{1cm} (5.26)

**Lemma 5.6.3.** If $(A_t) \in \mathcal{A}$, then the limit $S_y(A_t)$ exists. In particular,

(a) $S_y(P_t) = P_1$, $S_y(P^t) = P_1$;

(b) $S_y(cI) = c(-\infty)\chi_- + c(+\infty)\chi_+$ for $c \in PC(\hat{\mathbb{R}})$;

(c) $S_y(W^0(a)) = a(y^-)W^0(\chi_-) + a(y^+)W^0(\chi_+)$ for $a \in PC(\hat{\mathbb{R}})$;

(d) If $(j_t) \in J_0$ or $(j_t) \in J_1$, then $S_y(j_t) = 0$.

**Proof:** Assertion (a) is proved by first noting that, as the exponential is a uniformly continuous function, we have by Lemma 5.5.1 that $U_y P^t U_y - P^t$ goes in the norm to zero for any monotonically increasing sequence in $\mathbb{T}'$. Then as $Z_{-1} P^t Z_{-1}$ is simply the Galerkin projection with all mesh points divided by $\tau = \log(n_i)$, the result follows. Assertions (b) and (c) are consequences of Proposition 2.3.2. To prove the last one first remember that, due to the restrictions imposed on $\mathbb{T}'$, the value $\tau$ goes to infinity for any monotonically increasing sequence. Then note that for $G_\tau \in \mathcal{G}$, $\|S_{s\tau}(G_\tau)\| \leq \|G_\tau\| \to 0$. Also $U_{-y} Z_{-1} P^t Z_{-1}$ and $R_{-y} U_{-y} Z_{-1} P^t$ tend weakly to zero as $\tau \to \infty$, and as $Z_{-1} U_s$ and $Z_{-1} U_s R_{-y}$ are uniformly bounded, the result follows (see Lemmas 2.1.5, 2.1.6, 2.1.8 and Proposition 2.3.3).

**Proposition 5.6.4.** The local algebra $\mathcal{A}_{J,\infty,\infty}$, $y \in \mathbb{R}$ is isomorphic to the algebra $\mathcal{B} = \text{alg}(I, P_1, \chi_+, W^0(\chi_+))$ and this isomorphism is given by $S_y$.

**Proof:** This proof runs parallel to the proof of Proposition 3.4.8 in the finite section case. We only remark that $\Phi_{\infty,\infty}(P^t) = \Phi_{\infty,\infty}(P_t - P_t Q^t) \Phi_{\infty,\infty}(W^0(g))$ with $g$ continuous such that $g(y) = 1, g(\infty) = 0$, and $\Phi_{\infty,\infty}(P_t Q^t W^0(g)) = 0$ by Lemma 5.6.1 which means that $\Phi_{\infty,\infty}(P^t) = \Phi_{\infty,\infty}(P_t)$.

We are only left with the algebra at infinity.

5.7 The local Algebra $\mathcal{A}_{\infty,\infty}$

Here we will restrict our algebra, by dropping the coset containing the finite section projection. But first we will enumerate the generators of $\mathcal{A}_{\infty,\infty}$.

**Lemma 5.7.1.** The local algebra $\mathcal{A}_{\infty,\infty}$ is generated by the cosets $\Phi_{\infty,\infty}(I)$, $\Phi_{\infty,\infty}(\chi_+)$, $\Phi_{\infty,\infty}(W^0(\chi_+))$, $\Phi_{\infty,\infty}(P_1)$ and $\Phi_{\infty,\infty}(P^t)$.
So consider now the unital subalgebra $A_{\infty,\infty}^J$ sub of $A_{\infty,\infty}^J$ generated by the cosets $\Phi_{\infty,\infty}^J(I)$, $\Phi_{\infty,\infty}^J(\chi_+)$, $\Phi_{\infty,\infty}^J(W^0(\chi_+))$ and $\Phi_{\infty,\infty}^J(P^t)$. As in the case of the finite section treated before, this algebra has a non trivial center, because the coset $\Phi_{\infty,\infty}^J(\chi_+)$ commutes with the other generators. We can localize and obtain algebras $A_{\infty,\infty}^J_{\pm}$, which are generated by two projections (plus the identity) and the canonical homomorphism $\Phi_{\infty,\infty}^J_{\pm}$ from $A_J$ onto $A_{\infty,\infty}^J_{\pm}$. We introduce now the applications that to a sequence $(A_t)$ of $A$ associate

$$S_{\infty t}^\pm(A_t) = \tilde{V}_{\pm \log(\frac{n}{i})} A_t \tilde{V}_{\pm \log(\frac{n}{i})},$$

and the strong limit

$$S_{\infty}^\pm(A_t) = \text{s-lim}_{t \in \mathbb{T}} S_{\infty t}^\pm(A_t)$$

if it exists.

**Lemma 5.7.2.** The strong limits $S_{\infty}^\pm(A_t)$ exist for all $(A_t) \in A$. In particular,

(a) $S_{\infty}^\pm(\chi_+) = I$

(b) $S_{\infty}^\pm(P_t) = \chi_+$, $W_{\infty}^\pm(P^t) = \chi_+$;

(c) $S_{\infty}^\pm(cI) = c(\pm \infty)I$;

(d) $S_{\infty}^\pm(W^0(a)) = W^0(a)$;

(e) if $(j_t) \in A \cap J$, then $S_{\infty}^\pm(j_t) \in K$.

**Proof:** The only assertion that is not proved in the respective lemma for the finite section case is the second part of (b). For this it is only necessary to remark that $\tilde{V}_{\pm \log(\frac{n}{i})} P^t \tilde{V}_{\pm \log(\frac{n}{i})}$ is a galerkin projection with mesh $\Delta^t \mp \log(\frac{n}{i})$.

Using these homomorphisms (due to it’s contraction and Lemma 5.7.2) it is not difficult to see that $S_{\infty}^\pm$ are indeed homomorphisms, as in the case of the finite section method, one can see that

$$\sigma_{A_{\infty,\infty}^J_{\pm}}(\Phi_{\infty,\infty}^J_{\pm}(P^t W^0(\chi_+)) P^t)) = [0, 1]$$

and so it is possible to apply the Two Projections theorem. This theorem gives us symbol mappings $N_{\pm}$, to a space of matrix functions,

$$N_{\pm}(\Phi_{\infty,\infty}^J_{\pm}(\chi_+)) = e^t : s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$N_{\pm}(\Phi_{\infty,\infty}^J_{\pm}(P^t)) = p_1^t : s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};$$

$$N_{\pm}(\Phi_{\infty,\infty}^J_{\pm}(W^0(\chi_+)) = p_2^t : s \mapsto \begin{bmatrix} s & \sqrt{s(1-s)} \\ \sqrt{s(1-s)} & 1-s \end{bmatrix};$$

with $s \in [0, 1]$.
So define the homomorphism,
\[ M^\pm = N^\pm \Phi_{\infty,\infty}^J. \]

We just proved the following result:

**Proposition 5.7.3.** The local algebras \( A_{J,\infty,\infty}^{J,\pm} \) are isomorphic to the unital \( C^* \)-subalgebra of the algebra of 2 by 2 matrix functions defined on \([0, 1]\), which are diagonal at \( \{0, 1\} \) (and which is generated by \( e^t, p_1^t \) and \( p_2^t \)), and these isomorphisms are given by \( N^\pm \). The coset \( \Phi_{\infty,\infty}^J(A_\infty) \in A_{J,\infty,\infty}^{J,\pm} \) is invertible in the local algebra \( A_{J,\infty,\infty}^{J,\pm} \) if and only if \( M^\pm(A_t) \) are invertible. In particular
\[
M^\pm(cI) : s \mapsto \begin{bmatrix} c(\pm\infty) & 0 \\ 0 & c(\pm\infty) \end{bmatrix};
\]
\[
M^\pm(P^t) : s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};
\]
\[
M^\pm(W^0(a)) : s \mapsto a(-\infty) \begin{bmatrix} 1 - s & -\sqrt{s(1-s)} \\ -\sqrt{s(1-s)} & s \end{bmatrix} + a(+\infty) \begin{bmatrix} s & \sqrt{s(1-s)} \\ \sqrt{s(1-s)} & 1 - s \end{bmatrix}.
\]

### 5.8 Wiener-Hopf operators

Remember the Wiener-Hopf operator \( W(a) = \chi_+ W^0(a) \chi_+ + \chi_- \) and let now \( A \) be the algebra generated by the sequences
\[
(P^t W(a) P^t + Q^t)
\]
with \( a \in PC(\mathbb{R}) \). As usual let \( A^J \) be the quotient algebra \( (A + J)/J \). As in Lemmas 3.4.6 and 5.6.2 it is possible to see that the cosets \( (P^t W(g) P^t + Q^t) + J \) with \( g \in C(\mathbb{R}) \) are in the center of \( A^J \). Localizing over the maximal ideal space of the central subalgebra generated by these cosets (which is isomorphic to \( \mathbb{R} \)), one obtains for \( y \in \mathbb{R} \) local algebras that can be identified with the help of the homomorphism \( S_y \) defined in (5.26). For the algebra at the point \( \infty \), we obtain that it is generated by the cosets \( \Phi_\infty^J(\chi_- P^t + Q^t) \), \( \Phi_\infty^J(P^t \chi_+ W(\chi_+) \chi_+ P^t) \) and the identity. This algebra is commutative. The first generator is a projection, and we will again localize over the subalgebra generated by these coset. It has only two maximal ideals, namely the complex multiples of \( \Phi_\infty^J(\chi_- P^t + Q^t) \) and the complex multiples of \( \Phi_\infty^J(I - \chi_- P^t - Q^t) \). Thus we obtain two local algebras, \( A_{J,\infty,\infty}^{J,\pm} \) both of them singly generated. All it is left to do is find the spectrum of the generator \( \Phi_\infty^{J,\pm}(P^t \chi_+ W(\chi_+) \chi_+ P^t) \). This spectrum is equal to \([0, 1]\), and this is proved in the way described before Proposition 3.4.11. We can then conclude with the following result, which was already enunciated without details in [40], and that generalizes Elshner’s results in [13] regarding Galerkin methods for Wiener-Hopf operators with graded meshes (see the final remarks) and discontinuous polynomial spline spaces.
**Theorem 5.8.1.** For $a \in PC(\mathbb{R})$ the method

$$P^tW(a)P^tu = P^tv$$

to solve the equation $W(a)u = v$, $u, v \in L^2(\mathbb{R})$ applies if and only if the operator $W(a)$ and the operator $W(\tilde{a})$ (with $\tilde{a}(x) = a(-x)$) are invertible.

Note that for scalar Wiener-Hopf operators, the invertibility of $W(a)$ is equivalent to the invertibility of $W(\tilde{a})$.

### 5.9 Final remarks

This last result is still true if instead of a specific mesh $\Delta^l_q$, one considers any mesh of the class $\mathcal{M}$ or $\mathcal{M}_q$ as defined in [13]. For the meshes of class $\mathcal{M}$, as the largest length of the interval between mesh points tends to zero, the proofs are simpler. In any case, if the objective is to study only approximation methods for Wiener-Hopf operators, the algebras $\mathcal{F}$ and $\mathcal{A}$ are simpler because they have only to include operators of the type of (5.33) and they can be considered as acting in $\chi_+ \text{Im}(P^t)$. Also, it is possible to consider using this method the non-scalar case, which would give the same result.

For a mesh of the type of $\Delta^u_{q,2}$ the results would have been a little different. It is possible in that case to find an isomorphism from the local algebra $\mathcal{A}_0^J$ to an operator algebra. But in this algebra, the corresponding operator to $\Phi_0^J(P^t)$ is the (constant) projection associated with the mesh

$$\{\text{sgn}(j)|j|^q, \ -\infty < j < +\infty\}$$

and for whose appearance together with the singular integral operator invertibility conditions are not known.
Bibliography


Selbständigkeitsklärung

Hiermit erkläre ich, daß ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.


Pedro Santos
Theses
of the dissertation

Approximation methods for convolution operators on the real line by Pedro Alexandre Simões dos Santos

(a) This work is concerned with the applicability of several approximation methods, like the finite section method, Galerkin and collocation methods with maximum defect (discontinuous) splines, for uniform or non uniform meshes, to operators belonging to the closed subalgebra of $\mathcal{L}(L^2(\mathbb{R}))$ generated by operators of multiplication by piecewise continuous functions in $\mathbb{R}$ and convolution operators also with piecewise continuous generating functions. The traditional techniques are not powerful enough to give an answer to this problem. So the strategy followed is to use Banach algebra techniques.

(b) The algebra $\mathcal{E}$ of all uniformly bounded sequences of operators acting in $L^2(\mathbb{R})$ is introduced. The problem of applicability of a determined numerical method to a bounded operator acting in $L^2(\mathbb{R})$ is seen to be equivalent to a problem of invertibility in the algebra $\mathcal{E}$ factored by the sequences that tend in the norm to zero.

(c) This algebra is too large for an effective direct treatment, so the process of essentialization is used. This process consists in finding some ideals and homomorphisms which transform the invertibility problem in the initial algebra into a invertibility problem in a smaller algebra. A subset of this smaller algebra is taken, one which contains the elements corresponding to the specific numerical processes and operators under study. However, parting away from earlier works treating the real line case, the author here considered the projection sequence\(^3\) related to the approximation methods under study as an independent element of the algebra and the other elements to be the constant sequences of the operators that were to approximated.

(d) This much larger algebra is seen nevertheless to have a large center, due to the fact that in the essentialization step, a new type of lifting ideal was used. It is possible then to use localization techniques in order to find invertibility criteria for the elements of the algebras under study. The resulting local algebras are usually generated by 3 or even more idempotents. As a general theory for algebras generated by 3 or more generators does not exist, it is necessary, case by case, or find an isomorphism between the local (sequence) algebras and known (usually operator) algebras, or find enough relationships between the different generators in order to apply again localization procedures and/or a flip elimination scheme to finally arrive at algebras generated by one element or two projections. For all local algebras where this last procedure is used, it is then necessary to calculate some (non trivial) local spectrum of a combination of generators.

\(^3\)correctly speaking, at this stage we are dealing with cosets
(e) The process described above is initially used for the finite section method. The results are interesting in the fact there is a non-trivial relationship between Fredholmness of the operator restricted to the interval $[-1, 1]$ and applicability of the finite section method to that operator.

(f) When applied to maximum defect spline spaces with equidistant partitions, the results introduce a new operator, an infinite dimensional spline projection with unital (integer) partition. The Fredholmness of the initial operator when restricted to the image of that projection is also a necessary condition to the applicability of the numerical method.

(g) For non-uniform partitions the results obtained were not as general as before, but it was possible nevertheless to generalize to Wiener-Hopf operators with piecewise continuous symbol, some of the results obtained earlier by J. Elschner regarding piecewise polynomial spline spaces with maximum defect.
Lebenslauf

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